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CARDINALITIES OF BASES

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Most definitions of bases are specializations of the following one:

Given a class $\mathscr{V}$ of subsets of a set R and a transitive relation $<$ on R, call a subset of the union $\bigcup\mathscr{V}$ of $\mathscr{V}$ a *$\mathscr{V}$-base* iff it contains, for each V in $\mathscr{V}$ and v in V, some $b < v$ which belongs to $\bar{V} = \{r \in \bigcup\mathscr{V} : \exists v' < r, v' \in V\}$.

In this paper we relate the least power $w\mathscr{V}$ of $\mathscr{V}$-bases (assumed to be >0) to two other cardinalities depending on a relation σ defined on $\bigcup\mathscr{V}$, and state a few applications. One of these cardinalities, denoted by $\operatorname{cel}\sigma$, is the sup of the powers of all $E \subset \bigcup\mathscr{V}$ such that $a, b \in E$ and $a\sigma b$, $b\sigma a$ imply $a = b$. To define the other, call a class $\mathscr{G}$ of subsets of $\bigcup\mathscr{V}$ a *σ-grading of $\mathscr{V}$* iff its union contains, for each V in $\mathscr{V}$ and v in V, some $b < v$, $b \in V$, and for each F in $\mathscr{G}$ there is G in $\mathscr{G}$ with the following property: for each V in $\mathscr{V}$ and f in $F \cap V$ there is g in $G \cap V$ such that $f \geqq g'$ for each g' in G with $g\sigma g'$. Let $\sigma\mathscr{V}$ be the least power of such σ-gradings.

Theorem 1. *Let σ be any relation on $\bigcup\mathscr{V}$ such that $a\sigma b$ holds whenever there is V in $\mathscr{V}$ and v in V with $v < a \in V$ and $v < b$, and let $\bar{\mathscr{V}} = \{\bar{V} : V \in \mathscr{V}\}$. Then $w\mathscr{V}$ is finite iff both $\operatorname{cel}\sigma$ and $\sigma\bar{\mathscr{V}}$ are finite (which is sure if both $\operatorname{cel}\sigma$ and $\sigma\mathscr{V}$ are finite); otherwise*

$$w\mathscr{V} = \max(\operatorname{cel}\sigma, \sigma\bar{\mathscr{V}}) \leqq \max(\operatorname{cel}\sigma, \sigma\mathscr{V}).$$

Corollary. *For infinite $w\mathscr{V}$ the Suslin Property $w\mathscr{V} = \operatorname{cel}\sigma$ holds iff $\sigma\bar{\mathscr{V}} \leqq \operatorname{cel}\sigma$ and is implied by $\sigma\mathscr{V} \leqq \operatorname{cel}\sigma$.*

Subsequently let X be a topological space and wX its weight. Some applications of Theorem 1 to the determination of wX follow.

Direct applications. Let R be the class of all nonvoid subsets of X, $V(x)$ the class of all open neighbourhoods of $x \in X$, $\mathscr{V}$ the family of all $V(x)$ and $<$ the inclusion on R. Then $\bigcup\mathscr{V}$ is the topology of X, $\bar{\mathscr{V}} = \mathscr{V}$, $w\mathscr{V} = wX$, and Theorem 1 yields information on the weight of X for each σ in an infinite family of relations which contains the Intersection Relation ϱ ($A\varrho B$ iff A intersects B). Since $\operatorname{cel}\varrho$ is the cellularity $\operatorname{cel}X$ of X, the Corollary sheds some light on the problem of Suslin. (This method can be extended to arbitrary coverings of a set.)

Indirect applications. For each x in X let $V(x)$ be a class of subsets of X each of which contains x. Call the family $\mathcal{V}$ of all these classes a *σ-system on X* iff σ is any relation on $\bigcup\mathcal{V}$ with the following properties:

(i) $A\sigma B$ holds whenever $\mathcal{V}$ contains some $V(x)$ such that x is in B and A is in $V(x)$;

(ii) there is a σ-grading $\mathcal{G}$ of $\mathcal{V}$ which is *full*, i.e., such that $\bigcup\mathcal{G} = \bigcup\mathcal{V}$ and every element of $\mathcal{G}$ contains members of every $V(x)$;

(iii) the class of all $A \subset X$ such that for every x in A there is B in $V(x)$ with $B \subset A$ is the topology of X.

Theorem 2. *For every full σ-grading $\mathcal{G}$ of any σ-system on X,*

$$wX \leqq \text{cel}\,\sigma \,.\, \text{card}\,\mathcal{G}\,.$$

This theorem yields the following specialization.

Theorem 3. *If X is a uniformizable space and uX the least weight of uniformities compatible with the topology of X, then $wX \leqq$ cel X . uX.*

Proofs and other applications will appear elsewhere.