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ON TOPOLOGICAL ENTROPY

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In this communication we introduce an abstract scheme including the topological entropy (see [1]) as well as the Kolmogoroff-Sinaj's entropy (see [2], [3]) and also some other invariants.

Let \( P \) be a set with a reflexive and transitive relation \( \leq \). Assume that on the set \( P \) an associative binary operation \( \vee \) is defined such that \( A \vee B \geq A \) and \( A \vee B \geq B \) for every \( A, B \in P \). Further let \( T : P \rightarrow P \) and \( H : P \rightarrow (0, \infty) \) be any functions satisfying the following conditions:

1. \( H(\bigvee_{i=0}^{k} T^i(A)) \leq H(\bigvee_{i=0}^{j} T^i(A)) + H(\bigvee_{i=j+1}^{k} T^i(A)) \).
2. \( T(A \vee B) = T(A) \vee T(B) \).
3. \( H(T(A)) \leq H(A) \).

**Lemma.** Under these assumptions \( \lim n^{-1} \sum_{i=0}^{n-1} H(\bigvee_{i=0}^{n-1} T^i(A)) \) exists for any \( A \in P \).

**Definition.** For any given \( P, T, H \) and \( A \in P \) let us put \( h(A, T) = \lim n^{-1} \sum_{i=0}^{n-1} H(\bigvee_{i=0}^{n-1} T^i(A)) \), \( h(T) = \sup \{ h(A, T); A \in \bar{P} \} \); \( h(T) \) is called the entropy of the triple \( (P, T, H) \).

**Examples.**

1. **Topological entropy.** Let \( X \) be a topological space, \( f : X \rightarrow X \) a continuous map, \( P \) the family of all finite open coverings of \( X \) (\( R_1 \leq R_2 \) iff \( R_2 \) is a refinement of \( R_1 \)), \( H(A) = \log \text{card } A \), \( T(A) = f^{-1}(A) \).

2. **Kolmogoroff-Sinaj's entropy.** Let \( (X, S, m) \) be a probability measure space, \( f : X \rightarrow X \) a measure preserving transformation, \( P \) the family of all finite measurable decompositions \( A \) of \( X \) such that \( A, f^{-1}(A), \ldots, f^{-k}(A) \) are independent for all \( k \), \( T(A) = f^{-1}(A) \), \( H(A) = -\sum m(E) \log m(E); E \in A \).

3. **Entropy of an automorphism of a Boolean algebra.** Let \( B \) be a Boolean algebra, \( f \) an automorphism of \( B \). Let \( P \) be the set of all finite decompositions of the greatest element of \( B \). For \( A \in P \) put \( H(A) = \log \text{card } A \), \( T(A) = f(A) \).
Usually, if "two systems are isomorphic" then their entropies are equal. In general, two triples \((P, T, H)\) and \((R, S, G)\) are equivalent, if there is a bijection \(U : P \rightarrow R\) with the following properties:

1. \(U(A \vee B) = U(A) \vee U(B)\).
2. \(T \circ U = U \circ S\).
3. \(G(U(A)) = H(A)\).

**Theorem 1.** If \((P, T, H)\) and \((R, S, G)\) are equivalent then their entropies are equal.

We shall illustrate the preceding fact by the following three examples; the first two examples are well-known, the third one leads to a new result.

Let \(X_n\) be the set of all sequences \(x = \{x_i\}_{i=-\infty}^{\infty}\) of integers \(0, 1, \ldots, n - 1\). The shift is the map \(f : X_n \rightarrow X_n\) defined by the formula \(f(\{x_i\}_{i=-\infty}^{\infty}) = \{y_i\}_{i=-\infty}^{\infty}\), where \(y_i = x_{i+1}\) for every \(n\). There are at least three natural structures on \(X_n\):

1. Topology \(T_n\) with the subbase consisting of all cylinders \(\{x; x_i = j\}\) and the shift \(f\). It was proved in [1] that the topological entropy \(h(f) = \log n\). It follows that there is no homeomorphism \(g : X_n \rightarrow X_m (n \neq m)\) commuting with the shifts.

2. The (Bernoulli) dynamical system \((X_n, S_n, \mu, f)\); here \(S_n\) is the \(\sigma\)-algebra generated by the cylinders; \(\mu = \bigotimes_{i=-\infty}^{\infty} \mu_i\) is the Cartesian product of probability measures \(\mu_i\); for all \(i\), \(\mu_i = \mu_0\) and \(\mu_0\) is defined by means of \(n\)-tuple \((p_0, p_1, \ldots, p_{n-1})\), i.e., \(\mu_0(i) = p_i\). \(f\) is the shift. It is well-known that the Kolmogorov-Sinaj's entropy \(h(f) = -\sum p_i \log p_i\). Hence two Bernoulli systems with different entropies cannot be isomorphic. (Recently D. Ornstein [4] has proved the converse theorem.)

3. \(\sigma\)-algebras \(S_n\) generated by the cylinders and the automorphism \(f\) induced by the shift. Problem: Is there an isomorphism \(g : S_n \rightarrow S_m\) commuting with the shifts?

**Theorem 2.** If \(S_n\) is the \(\sigma\)-algebra generated by the cylinders, \(f\) is the automorphism of \(S_n\) generated by the shift and \(h(f)\) is the entropy introduced in the third example, then \(h(f) = \log n\).

**Corollary.** Given \(n \neq m\), there is no isomorphism \(g : S_n \rightarrow S_m\) commuting with the shifts.

The last corollary was proved also in [5], but in another way.

Of course, also some further theorems can be proved in the general case. So \(h(T^n) = k h(T)\), \(h(T_1 \times T_2) = h(T_1) + h(T_2)\) and if \(A \in P\) is an element such that \(\bigvee_{i=0}^{n-1} T^i(A)\) "generates" the set \(P\), then \(h(T) = h(T, R)\).
Finally we list further examples satisfying the assumptions of our scheme:

4. Another type of topological entropy. Let $P$ be the family of all open coverings of $X$ having refinements of finite orders, $H(A) = \log \min \{\text{order } B; B \text{ is a refinement of } A\}$. This invariant probably corresponds to the topological dimension. If $X$ is a topological space of finite dimension, then $\dim X \geq e^{H(T)} - 1$.

5. Group endomorphism entropy (see [1]). Let $G$ be an Abelian group, $P$ the family of all finite subgroups, $A \leq B$ iff $A \subset B$, $T$ an endomorphism and $H(A) = \log \text{order } A$.

6. Entropy of a measure preserving transformation. Let $P$ be a ring of sets (ordered by the inclusion), $H$ a measure on $P$, $T$ a measure preserving transformation.

7. Entropy of an operator. $P$ is the system of all integrable functions (ordered as usually), $H$ is the integral, $T(f) = f + g$ where $g$ is a fixed non-positive function.

References


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