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REALIZATIONS OF CLOSURE SPACES BY SET SYSTEMS

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Let $\mathscr{C}$ be the category of all topological spaces in the sense of [1] with continuous mappings as morphisms, i.e., an object of $\mathscr{C}$ is a pair $\langle P, u\rangle$ where $u : \exp P \to \exp P$ fulfils the following axioms:

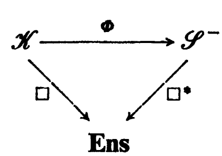
$$u\emptyset = \emptyset\,, \quad X \subset P \Rightarrow X \subset uX\,, \quad X \subset Y \subset P \Rightarrow uX \subset uY\,.$$

Let us remind that $f : P \to Q$ is a continuous mapping from $\langle P, u\rangle$ into $\langle Q, v\rangle$ iff $f(uX) \subset vf(X)$ for all $X \subset P$. Similarly, a map $f : P \to Q$ is called inversely continuous if $f(uX) \supset vf(X)$. If $f(uX) = vf(X)$, the map f is called closed. $\mathscr{C}'$ or $\mathscr{C}''$ will be the categories with the same objects as $\mathscr{C}$ has, but with inversely continuous mappings or closed mappings respectively as morphisms.

$\mathscr{A}$ will be the full subcategory of $\mathscr{C}$ formed by all topological spaces $\langle P, u\rangle$ from $\mathscr{C}$ for which $u(X \cup Y) = uX \cup uY$ for all X, $Y \subset P$, (the theory of such spaces is developed in [2]), and with continuous mappings as morphisms. $\mathscr{B}$ means the full subcategory of $\mathscr{A}$ formed by all topological spaces $\langle P, u\rangle$ from $\mathscr{A}$ with $u(uX) = uX$ for all $X \subset P$ (morphisms – continuous mappings). Let $\mathscr{A}'$, $\mathscr{A}''$, $\mathscr{B}'$, $\mathscr{B}''$ be defined in similar way as $\mathscr{C}'$, $\mathscr{C}''$ are for $\mathscr{C}$.

Now, let $\mathscr{S}^-$ be the category defined in the following way. Objects of $\mathscr{S}^-$ are pairs $\langle P, S\rangle$ where P is a set and $S \subset \exp P$. Morphisms from $\langle P, S\rangle$ into $\langle Q, T\rangle$ are mappings $f : P \to Q$ for which $X \in T \Rightarrow f^{-1}(X) \in S$. If instead of this condition $X \in S \Rightarrow f(X) \in T$ holds we get the category $\mathscr{S}^+$. $\mathscr{S}$ will mean the intersection of $\mathscr{S}^-$ and $\mathscr{S}^+$.

A full embedding of one category into another is defined as in [3], i.e., it is a full functor true for morphisms and objects. If $\mathscr{K}$ means some of categories $\mathscr{C}$, $\mathscr{A}$, $\mathscr{B}$ or their subcategories then by realization of $\mathscr{K}$ in $\mathscr{S}^-$ such a full embedding Φ is meant for which



where **Ens** is the category of all sets together with all mappings as morphisms and $\square$, $\square^*$ are forgetful functors $[\square\langle P, u\rangle = P = \square^*\langle P, S\rangle]$ (see e.g. [4]).

It is clear that the systems of open (closed) sets for an object from $\mathscr{B}$ induce two realizations Φ_1, Φ_2 of $\mathscr{B}$ into $\mathscr{S}^-$, similarly systems of closed sets induce the realizations Φ_3, Φ_4 of $\mathscr{B}'$ into $\mathscr{S}^+$, or $\mathscr{B}''$ into $\mathscr{S}$ respectively. It is not difficult to prove (probably this is a known result) the following

Proposition 1. *Let Φ be a full embedding of $\mathscr{B}$ into $\mathscr{S}^-$ such that if $\Phi\langle X, u\rangle = \langle Y, S\rangle$ then $\emptyset, Y \in S$. Then Φ is equivalent either to Φ_1 or to Φ_2.*

An immediate consequence is

Proposition 1'. *Φ_1, Φ_2 are the only realizations of $\mathscr{B}$ into $\mathscr{S}^-$.*

Similarly one can prove the following

Proposition 2. *Φ_3 is the only realization of $\mathscr{B}'$ into $\mathscr{S}^+$*

Up to now, for $\mathscr{B}''$, the authors can only deduce that for every realization $\Phi : \mathscr{B}'' \to \mathscr{S}$ for all $\langle P, u\rangle$ the following assertion is valid: If $\Phi\langle P, u\rangle = \langle P, S\rangle$ and X is a closed set in $\langle P, u\rangle$, then $X \in S$.

It is a natural question to ask what can be said about realizations of $\mathscr{C}$ or $\mathscr{A}$ in $\mathscr{S}^+$. A detailed investigation of the system of three-point spaces of $\mathscr{A}$, and not quite trivial transfer of some results to infinite case, give the following negative answers.

Proposition 3. *Let $\mathscr{K}$ be a full subcategory of $\mathscr{A}$ ($\mathscr{A}'$) for which there exists a set X, $\operatorname{card} X \geq 3$, such that all topological spaces from $\mathscr{A}$ of the form $\langle X, u\rangle$ are objects of $\mathscr{K}$. Then there is no realization of $\mathscr{K}$ into $\mathscr{S}^-$ ($\mathscr{S}^+$).*

Proposition 4. *The analogous assertion to that in Proposition 3 is valid for $\mathscr{A}''$ and $\mathscr{S}$ under the condition that $\operatorname{card} X = 3$ or X is infinite.*

It remains an open question, whether finite X with $\operatorname{card} X \geq 4$ can be allowed, too[1]).

There exist, of course, embeddings of $\mathscr{C}$ in $\mathscr{S}^-$ defined by various set theoretical functors. E.g., one can prove

Proposition 5. *Let $\langle P, u\rangle$ be an object in $\mathscr{C}$. Put $\mathscr{S}_{\langle P, u\rangle}(\emptyset) = \{\emptyset\}$, $\mathscr{S}_{\langle P,u\rangle}(M) = \{\exp P\}$ for all $M \subset P$ with $uM = P$, $\mathscr{S}_{\langle P,u\rangle}(M) = \{\{X \cup Y \mid X \in \mathscr{N}, Y \subset M, Y \neq \emptyset\} \mid \mathscr{N} \in \exp\exp(P - uM), \mathscr{N} \neq \emptyset\}$ otherwise. Let $\mathscr{S}\langle P,u\rangle = \bigcup_{M \subset P} \mathscr{S}_{\langle P,u\rangle}(M)$. Put*

$$\Phi\langle P, u\rangle = \langle \exp P, \mathscr{S}\langle P, u\rangle\rangle .$$

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[1]) Added in proofs. The answer is positive.

For a continuous map $f : \langle P, u\rangle \to \langle Q, v\rangle$ *put* $\Phi(f) : \exp P \to \exp Q$ *with* $\Phi(f)(X) = f(X)$ *for all* $X \subset P$. *Then* Φ *is an embedding of* $\mathscr{C}$ *in* $\mathscr{S}^-$.

Or

Proposition 6. *Let* $\langle P, u\rangle$ *be an object in* $\mathscr{C}$. *For every* $x \in P$ *and every neighborhood* $\mathscr{U}$ *of* x *in* $\langle P, u\rangle$, *put* $\langle x, \mathscr{U}\rangle = \{\langle x, y\rangle \mid y \in \mathscr{U}\}$. *Let* $\mathscr{S}\langle P, u\rangle$ *be an additive hull of the system of all* $\langle x, \mathscr{U}\rangle$. *For a continuous map* $f : \langle P, u\rangle \to \langle Q, v\rangle$ *define* $f \times f : P \times P \to Q \times Q$ *as usual by* $f \times f\langle x_1, x_2\rangle = \langle f(x_1), f(x_2)\rangle$. *Put*

$$\Phi\langle P, u\rangle = \langle P \times P, \mathscr{S}\langle P, u\rangle\rangle\,, \quad \Phi(f) = f \times f\,.$$

Then Φ *is an embedding of* $\mathscr{C}$ *in* $\mathscr{S}^-$.

(Notice that, if $\langle P, u\rangle$ is in $\mathscr{A}$, then $\Phi\langle P, u\rangle$ is, in fact, in $\mathscr{B}$.)

Proofs and other details of the above propositions will be published partly in a common paper, partly in the first author's Thesis.

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