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GAME-THEORETICAL APPROACH TO SOME MODIFICATIONS OF GENERALIZED TOPOLOGIES

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0. There is an interesting possibility of a use of generalized topologies in the theory of extensive games: In my works, I introduced the so-called SN-games [= simultaneous nondeterministic games; at each nonfinal position of an SN-game all the players play mutually independently (the simultaneousness), knowing the preceding course of play, but their common influence need not determine the next position uniquely (the (local) nondeterminateness)], and I have shown that a certain reduced description (see [5], § 2.30.2 etc.) of SN-games is sufficient for the introduction of analogues of usual game-theoretical notions, for defining certain significant classes of SN-games, and, among other, for proving various strong game-theoretical theorems. (Cf., e.g., [3], [5], [6].) (Such a conception of SN-games has also admitted the introduction of the corresponding "descriptive theory", see [3], § 9, or [5], § 7, and of topological games, see [4], or [5], § 8; of course, various known results concerning games with perfect information etc. are included as corollaries in theorems on SN-games.)

In this communication, we shall show that the game-theoretical interpretation of some partial unary operations (in particular, of those corresponding to some modifications) can be used for investigating their algebraic properties. In contradiction to [3]–[6], this paper contains no principal theorem: the goal is to show typical approaches, and to present a number of particular results obtained by means of them. A very short introduction of necessary auxiliary definitions, comments, and properties is given.

1.1. "$\mathscr{A} := \mathscr{B}$" means "$\mathscr{A}$ is defined to be equal to $\mathscr{B}$". $\operatorname{dom} f$ means the domain of the mapping f. P will be a set, $\exp P := \{A \mid A \subseteq P\}$, $\mathscr{T}(P) := (\exp P)^{\exp P}$; P will be usually fixed, and then we shall write, e.g., only $\mathscr{T}$ instead of $\mathscr{T}(P)$. Let $\mathbf{1}$, $(-\mathbf{1})$, $\underline{X}$ (where $X \subseteq P$) be such that $\mathbf{1}A = A$, $(-\mathbf{1})\,A = P - A$, $\underline{X}A = X$ for any $A \subseteq P$. $\mathscr{T}$ together with the usual composition of mappings forms a monoid (the identity $\mathbf{1}$ is its unit), and $\mathscr{T}$ together with the usual partial ordering $\leqq$ ($u \leqq v$ iff $uA \subseteq vA$ for any $A \subseteq P$) forms a complete lattice. $\leqq$ can be introduced in a somewhat different way, too: let Φ be the mapping for which $\operatorname{dom} \Phi = \mathscr{T}$, $\Phi(u) = \{(x, A) \mid A \subseteq P,\ x \in uA\}$ ($u \in \mathscr{T}$); then Φ is a one-to-one ("canonical") mapping of $\mathscr{T}$ onto $\exp(P \times \exp P)$, but the latter set is a complete lattice with respect to $\subseteq$,

$\bigcup$, $\bigcap$, and this complete lattice structure can be transformed by Φ^{-1} back onto $\mathcal{T}$ [thus, we shall write shortly, e.g., $u \subseteq v$, $u \cup v$ instead of $\Phi(u) \subseteq \Phi(v)$, $\Phi^{-1}(\Phi(u) \cup \cup \Phi(v))$, respectively]; it is easy to see that this (transformed) $\subseteq$ is the same as $\leqq$. (Besides, there is the canonical one-to-one mapping of $\mathcal{T}$ onto $(\exp \exp P)^P$.) Cf. [12], and [5], § 2a.

1.2. If $R(.), R_1(.), \ldots, R_k(.)$ are some (one-variable) propositional functions defined at least on $\mathcal{T}$, we denote $\mathcal{T}_R := \{u \mid u \in \mathcal{T},\ R(u)\}$, $\mathcal{T}_{R_1 \ldots R_k} := \mathcal{T}_{R_1} \cap \cap \ldots \cap \mathcal{T}_{R_k}$. The following propositional functions are often used (we write "R:" instead of "$R(u)$ iff"): R: $u\emptyset = \emptyset$; M: $A \subseteq B \subseteq P \Rightarrow uA \subseteq uB$; E: $\mathbf{1} \subseteq u$; I: $u \subseteq \mathbf{1}$; U: $u^2 = u$; A: $A_1, A_2 \subseteq P \Rightarrow u(A_1 \cup A_2) = uA_1 \cup uA_2$. Now, $\mathcal{T}_{\mathrm{REUA}}$ is the set of (all) *topologies* (on P). Various kinds of *generalized topologies* obtained by replacing the Kuratowski axioms by some weaker ones have been investigated (let us quote several of many papers concerning these problems: [1], [7]–[14]; especially, closure spaces ($\mathcal{T}_{\mathrm{REA}}$, see [2], Sec. 14) and *Čech topological spaces* ($\mathcal{T}_{\mathrm{REM}}$ – in the sense of [1]; of course, $\mathcal{T}_{\mathrm{A}} \subseteq \mathcal{T}_{\mathrm{M}}$) have been often investigated (in particular, the problems of modifying Čech topologies and the properties of the constellations of Čech topologies have been studied circumstantially; see, e.g., [1], [8], [9], [11]), and some topological considerations have been performed even for general elements of $\mathcal{T}$ itself (*Koutský topologies*, or "topologies without axioms", see [7], [13], and a part of [14]).

In this communication, we shall consider another kind of generalized topologies, namely $\mathcal{T}_{\mathrm{RM}}$; they will be called *game topologies* (on P), cf. [5], §§ 2.8, 2.26.3. Under a *type* we shall mean (P, P_0) where $P_0 \subseteq P$ (in [5], § 1.1, $P \neq \emptyset$ was supposed besides); $\mathcal{T}_{\mathrm{RM}}(P, P_0) := \{u \mid u \in \mathcal{T}_{\mathrm{RM}}(P),\ uP = P - P_0\}$ will be the set of game topologies of the type (P, P_0). ([5], § 2.13 etc.)

1.3. Under a *game system* (on P) we shall mean a non-empty system $\mathcal{U} = (u_j)_{j \in J} \in (\mathcal{T}_{\mathrm{RM}})^J$ such that all the u_j have the same type, and $\bigcap_{j \in J} A_j = \emptyset$ implies $\bigcap_{j \in J} u_j A_j = \emptyset$ for every $(A_j)_{j \in J} \in (\exp P)^J$. [For an SN-game $\mathcal{G}$, let P (P_0, J) be the set of its positions (final positions, players, respectively), let (for $j \in J$) $u_j \in \mathcal{T}(P)$ be such that $u_j A = \{x \mid x \in P$, in $\mathcal{G}$, j can guarantee at x that the next position will (exist and) belong to $A\}$ for any $A \subseteq P$, let $Z := P - P_0$; then $\mathcal{U} := (u_j)_{j \in J}$ is a game system, $u_j P = Z$ for each $j \in J$, and Z is the set of nonfinal positions of $\mathcal{G}$. On the other hand, if some $\mathcal{U}$ is a game system, then $\mathcal{U}$ can be obtained in the above mentioned way, to a suitable $\mathcal{G}$. See [5], § 2.29–30, § 2 (47), (26) etc.]

1.4. Under an *operator* we shall mean a partial unary operation in $\mathcal{T}$, i.e., a mapping of a subset of $\mathcal{T}$ into $\mathcal{T}$. We shall need, in particular, these operators: $u \to u^k$ ($k = 0, 1, \ldots$), defined as the kth power in the monoid $\mathcal{T}$; $u \to \tilde{u} := \underline{P - uP} \cup \cup u$; $u \to \bar{u} := (-\mathbf{1}) . u . (-\mathbf{1})$; $u \to u' := \underline{uP} \cap \bar{u}$ (i.e., $u'A = uP - u(P - A)$); $u \to \check{u}$ where $\check{u}A = uA \cup (A - uP)$.

It holds: if $u \in \mathscr{T}_{\mathrm{RM}}(P, P_0)$ (where $P_0 \subseteq P$), then $u' \in \mathscr{T}_{\mathrm{RM}}(P, P_0)$, $u'' = u$; if $u_1 \in \mathscr{T}_{\mathrm{RM}}(P, P_0)$, then u' is the greatest (under $\subseteq$) element of $\{u_2 \mid u_2 \in \mathscr{T}_{\mathrm{RM}}, (u_1, u_2)$ is a game system$\}$. (See [5], § 4 (26)–(28), § 4.8.4 etc.) An SN-game $\mathscr{G}$ is said to be complete iff card $J = 2$ and $(u_{j_2})' = u_{j_1}$ for $\{j_1, j_2\} = J$ (where J, u_j etc. have the meaning given by the remark in Sec. 1.3). Complete games have very significant properties, similarly as their particular case – two-player Bergean games with perfect information. (Cf. [5], §§ 4d, 6c, and [6].)

1.5. For $R_1(.), \ldots, R_k(.)$ (Sec. 1.2), under the *upper* [*lower*] $R_1 \ldots R_k$-*modification* of u $(\in \mathscr{T})$ we mean $v \in \mathscr{T}_{R_1 \ldots R_k}$ such that (i) $u \subseteq v$ $[u \supseteq v]$, and (ii) $v \subseteq w$ $[v \supseteq w]$ whenever $w \in \mathscr{T}_{R_1 \ldots R_k}$ and $u \subseteq w$ $[u \supseteq w]$; of course, there exists at most one upper [lower] $R_1 \ldots R_k$-modification of u, thus the forming of upper [lower] $R_1 \ldots R_k$-modifications may be considered an operator. (Cf. [2], Sec. 31 B.)

Let $\mathbf{1} \nparallel u \in \mathscr{T}_{\mathrm{M}}$; the well-known construction of the transfinite powers of u can be expressed in such a way: $u^0 := \mathbf{1}$, $u^\xi := \lim\limits_{0 \leqq \eta < \xi} u \,.\, u^\eta$ ($\xi > 0$ is an ordinal number); we put formally $u^\infty := u^\xi$ for (any) ξ such that $u^\xi = u^{\xi+1}$ (∞ is not an ordinal number). It is known that u^∞ is the upper [lower] U-*modification* of u if $u \in \mathscr{T}_{\mathrm{EM}}$ $[u \in \mathscr{T}_{\mathrm{IM}}]$. For each $u \in \mathscr{T}_{\mathrm{M}}$, $\mathbf{1} \cup u$ $[\mathbf{1} \cap u]$ belongs to $\mathscr{T}_{\mathrm{EM}}$ $[\mathscr{T}_{\mathrm{IM}}]$, and $u^\triangle := (\mathbf{1} \cup u)^\infty$ $[u_\triangledown := (\mathbf{1} \cap u)^\infty]$ is the *upper* UEM-*modification* [*lower* UIM-*modification*] *of* u. The modifications $^\triangle$, $_\triangledown$, and also the operators $u \to u_\triangle := (\tilde{u})_\triangledown$, $u \to u^\triangledown := (\tilde{u})^\triangle$ (where $u \in \mathscr{T}_{\mathrm{M}}$; of course, then $\tilde{u} \in \mathscr{T}_{\mathrm{M}}$) are important in game considerations.

For $u \in \mathscr{T}_{\mathrm{M}}$ it holds: $\overline{u^\triangle} = (\bar{u})_\triangledown$, $\overline{u_\triangledown} = (\bar{u})^\triangle$, $u_\triangle = \overline{(u')^\triangle}$, $u^\triangledown = \overline{(u')_\triangledown}$; if, moreover, $u \in \mathscr{T}_{\mathrm{RM}}$, then $u^\triangle = \overline{(u')_\triangle}$, $u_\triangledown = \overline{(u')^\triangledown}$. ([5], § 5.18.)

2.1. To a given type (P, P_0) we introduce: $Z := P - P_0$, $\mathbf{Z} := \bigcup\limits_{0 \leqq l < \omega_0} Z^{\{0, \ldots, l\}}$, $\boldsymbol{S} := ((\exp P) - \{\emptyset\})^{\mathbf{Z}}$, and

$$\boldsymbol{P} := \bigcup_{0 \leqq l \leqq \omega_0} \{\boldsymbol{x} = (x_k \mid 0 \leqq k < 1 + l) \mid x_k \in Z \text{ if } 0 \leqq k < l,\ x_k \in P_0 \text{ if } k = l\},$$

where ω_0 is the first infinite ordinal number. For $\mathbf{z} = (z_0, \ldots, z_l) \in \mathbf{Z}$ we define $\kappa(\mathbf{z}) := z_l$, $l(\mathbf{z}) := l$. For $\boldsymbol{x} = (x_k \mid 0 \leqq k < 1 + l) \in \boldsymbol{P}$ we write shortly $\boldsymbol{x} = (x_k)$ and define $l(\boldsymbol{x}) := l$. For $\boldsymbol{\sigma} \in \boldsymbol{S}$, $x \in P$ we put

$$\mathrm{s}(\boldsymbol{\sigma}) := \{\boldsymbol{x} = (x_k) \mid \boldsymbol{x} \in \boldsymbol{P},\ x_{k+1} \in \boldsymbol{\sigma}(x_0, \ldots, x_k) \text{ if } 0 \leqq k < l(\boldsymbol{x})\},$$
$$\mathrm{s}(x, \boldsymbol{\sigma}) := \{\boldsymbol{x} = (x_k) \mid \boldsymbol{x} \in \mathrm{s}(\boldsymbol{\sigma}),\ x_0 = x\} \quad (\neq \emptyset).$$

2.2. Let $u \in \mathscr{T}_{\mathrm{RM}}(P, P_0)$ in this section. We put $\boldsymbol{S}(u) := \{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in (\exp P)^{\mathbf{Z}}, \kappa(\mathbf{z}) \in u(\boldsymbol{\sigma}\mathbf{z})$ for every $\mathbf{z} \in \mathbf{Z}\}$ $(\subseteq \boldsymbol{S})$. Further, let $\boldsymbol{u} \in (\exp P)^{\exp \boldsymbol{P}}$ be given by $\boldsymbol{u}\boldsymbol{A} := \{x \mid x \in P,\ \mathrm{s}(x, \boldsymbol{\sigma}) \subseteq \boldsymbol{A}$ for some $\boldsymbol{\sigma} \in \boldsymbol{S}(u)\}$. There holds (see [5], § 3a): $\boldsymbol{u}\emptyset = \emptyset$; $\boldsymbol{A} \subseteq \boldsymbol{B} \subseteq \boldsymbol{P} \Rightarrow \boldsymbol{u}\boldsymbol{A} \subseteq \boldsymbol{u}\boldsymbol{B}$; if $(u_j)_{j \in J}$ is a game system such that $u_j P = Z$ (cf. Sec. 1.3), then $\bigcap_{j\in J} \boldsymbol{A}_j = \emptyset$ implies $\bigcap_{j\in J} \boldsymbol{u}_j\boldsymbol{A}_j = \emptyset$ for any $(\boldsymbol{A}_j)_{j\in J} \in (\exp \boldsymbol{P})^J$ ([5], § 4.18). But if $v := u'$ (cf. Sec. 1.4), it may happen there is $\boldsymbol{A} \subseteq \boldsymbol{P}$ such that $\boldsymbol{vA} = \boldsymbol{uP} - \boldsymbol{u}(\boldsymbol{P} - \boldsymbol{A})$ does not hold ([5], §§ 4.21.5, 4.23–24); this very important fact concerns immediately some connections with axiomatic set theories. [In terms of the game interpretation mentioned in the remarks in Sec. 1.3, $\boldsymbol{P}$ ($\exp \boldsymbol{P}$) is the set of variants (aims) at $\mathscr{G}$, $S(u_j)$ is the set of player j's strategies; $x \in \boldsymbol{u}_j\boldsymbol{A}$ means that j can enforce $\boldsymbol{A}$ from x, etc. Cf. [5], §§ 2c, 3a.]

3.1. In Sec. 3, let (P, P_0) be a type, $u \in \mathscr{T}_{\mathrm{R\dot{M}}}(P, P_0)$, $v := u'$. For $\boldsymbol{p} \in (\exp \boldsymbol{P})^{\exp P}$ (= the set of *aim-mappings at* (P, P_0)) we define $\bar{\boldsymbol{p}} \in (\exp \boldsymbol{P})^{\exp P}$ by $\bar{\boldsymbol{p}}A = \boldsymbol{P} - \boldsymbol{p}(P - A)$ (cf. Sec. 1.4!). Similarly as in [5] (§ 5.8, or [3], too), we shall denote a general aim-mapping by symbol $\boldsymbol{p}^\varepsilon$ and we shall write $\boldsymbol{p}_\varepsilon := \overline{\boldsymbol{p}^\varepsilon}$ (this symbolism admits various concrete forms, e.g.: $\boldsymbol{p}^\varepsilon = \boldsymbol{p}^\triangle, \boldsymbol{p}_\square, \tilde{\boldsymbol{p}}, \boldsymbol{p}_\varepsilon = \boldsymbol{p}_\triangle, \boldsymbol{p}^\square, \underset{\sim}{\boldsymbol{p}}$, respectively). To such $\boldsymbol{p}^\varepsilon$ we define $u^\varepsilon := \boldsymbol{u} \,.\, \boldsymbol{p}^\varepsilon$; $^\varepsilon$ itself is then considered an operator (dom $^\varepsilon$ = $= \mathscr{T}_{\mathrm{RM}}(P, P_0)$), and $\boldsymbol{p}^\varepsilon$ is said to be the *aim-meaning of* $^\varepsilon$. $\boldsymbol{p}^\varepsilon$ and $^\varepsilon$ are said to be *normal* iff (cf. Sec. 2.2!) $(v^\varepsilon A =)\, \boldsymbol{v} \,.\, \boldsymbol{p}^\varepsilon A = \boldsymbol{uP} - \boldsymbol{u}(\boldsymbol{P} - \boldsymbol{p}^\varepsilon A)$ $(= P - \boldsymbol{u} \,.\, \boldsymbol{p}_\varepsilon(P - A) = P - u_\varepsilon(P - A) = \overline{u_\varepsilon}A)$, i.e., iff $(u')^\varepsilon = \overline{u_\varepsilon}$ (identically). In general, always $(u')^\varepsilon \subseteq \overline{u_\varepsilon}$, and $^\varepsilon$ is normal iff $_\varepsilon$ is normal.

3.2. Some operators (e.g., $u \to u^0$ $(= \mathbf{1})$) have simple normal aim-meanings. The union (intersection, product) of two operators having normal aim-meanings need not have an aim-meaning. But if $u \to u^\varepsilon$ is an operator having a [normal] aim-meaning, then $u \to u \,.\, u^\varepsilon$, $u \to \breve{u} \,.\, u^\varepsilon$, and $u \to \tilde{u} \,.\, u^\varepsilon$ have [normal] aim-meanings.

3.3. (Cf. [3], §§ 2.8.1, 6.9.4, or [5], part IV.) Let $\boldsymbol{p}^\varepsilon$, $\boldsymbol{p}_\varepsilon$ ($\varepsilon = \triangle, \nabla, \square$) be such that

$$x \in \boldsymbol{p}^\triangle A \; [x \in \boldsymbol{p}_\triangle A] \Leftrightarrow x_k \in A \text{ for some [each] } k\,,$$

$$\boldsymbol{p}^\nabla A = \boldsymbol{P}_{\mathrm{F}} \cup \boldsymbol{p}^\triangle A\,, \quad \boldsymbol{p}_\nabla A = (\boldsymbol{P} - \boldsymbol{P}_{\mathrm{F}}) \cap \boldsymbol{p}_\triangle A\,,$$

$$x \in \boldsymbol{p}^\square A \Leftrightarrow \text{for each } k \text{ there exists } r \geqq k \text{ such that } x_r \in A\,,$$

$$x \in \boldsymbol{p}_\square A \Leftrightarrow \text{there exists } k \text{ such that } x_r \in A \text{ for each } r \geqq k\,,$$

where

$$x = (x_k) \in \boldsymbol{P}\,, \quad A \subseteq P\,, \quad 0 \leqq k\,, \quad r < 1 + l(x)\,,$$

$$\boldsymbol{P}_{\mathrm{F}} := \{x \mid x \in \boldsymbol{P},\; l(x) < \omega_0\}\,;$$

it is easy to see that, indeed, $\boldsymbol{p}_\varepsilon = \overline{\boldsymbol{p}^\varepsilon}$ for $\varepsilon = \triangle, \nabla, \square$ (cf. Sec. 3.1).

Now, u^ε and u_ε are defined twice for $\varepsilon = \triangle, \nabla$ (and $u \in \mathscr{T}_{\mathrm{RM}}$), namely by Sec. 3.1 and 3.3, and by 1.5; nevertheless, it can be proved that these two definitions yield the same concepts. Moreover, it can be shown, among others, that $\boldsymbol{p}^\varepsilon$, $\boldsymbol{p}_\varepsilon$

($\varepsilon = \triangle, \triangledown, \square$) are normal. From the definitions it follows trivially that u^ε, $u_\varepsilon \in \mathscr{T}_{\mathrm{M}}(P)$, $u^\triangle\emptyset = \emptyset$, $u_\triangle P = P$, and $u_\triangledown \subseteq u_\triangle \subseteq u_\square \subseteq u^\square \subseteq u^\triangle \subseteq u^\triangledown$, $u_\triangle \subseteq \mathbf{1} \subseteq u^\triangle$, $u^\triangledown = u^\triangle . (\mathbf{1} \cup \underline{P_0'})$, $u_\triangledown = u_\triangle . (\mathbf{1} \cap \underline{Z})$, etc.

3.4. It holds $\breve{u} = \overline{\breve{v}} \in \mathscr{T}_{\mathrm{RM}}(P, \emptyset)$; we shall write (cf. Sec. 3.2!) $u^{\smile\varepsilon} := \breve{u} . u^\varepsilon$, $u_{\smile\varepsilon} := \breve{u} . u_\varepsilon$ (then $u_{\smile\varepsilon} = \overline{(u')^{\smile\varepsilon}}$ if $u_\varepsilon = \overline{(u')^\varepsilon}$).

Clearly, $(\breve{u})^\triangledown = (\breve{u})^\triangle$, $(\breve{u})_\triangledown = (\breve{u})_\triangle$. The following important equalities hold:

$$(\breve{u})^\varepsilon = u^\varepsilon\,, \quad (\breve{u})_\varepsilon = u_\varepsilon \quad \text{for} \quad \varepsilon = \triangle, \square\,,$$
$$u^\square = (u^{\smile\triangle})_{\smile\triangle}\,, \quad u_\square = (u_{\smile\triangle})^{\smile\triangle}\,.$$

(To derive the propositions presented in this paper, it is natural to use the latter two equalities to prove the normality of $^\square$ and $_\square$; nevertheless, there are other ways, cf. [3], § 6a, or § 7b, or § 9b.)

4.1. Theorem. *Let (P, P_0) be a type, $u \in \mathscr{T}_{\mathrm{RM}}(P, P_0)$. Then:*

A.

(1) $u^\triangledown . u^\varepsilon = u^\triangledown$ *for* $\varepsilon = \triangle, \triangledown$

(2) $u^\triangle . u^\varepsilon = u^\varepsilon$ *for* $\varepsilon = \triangle, \triangledown, \square$

(3) $u^\varepsilon . u_\square = u_\square . u_\square = u_\square$ *for* $\varepsilon = \triangle, \square$

(4) $u_\varepsilon . u^\triangle . u_\triangle = u^\triangle . u_\triangle$ *for* $\varepsilon = \triangle, \square$

(5) $u_\varepsilon . u^\triangle . u_\triangledown = u^\triangle . u_\triangledown$ *for* $\varepsilon = \triangledown, \triangle, \square$

(6) $u^\triangledown . u^\square . u^\triangledown = u^\square . u^\triangledown$

(7) $u_\triangledown . u^\square . u_\triangledown = u^\square . u_\triangledown$

(8) $u_\triangle . u^\triangledown . u^\square = u^\triangledown . u^\square$

(9) $u_\varepsilon . u^\triangledown . u_\triangle = u^\triangledown . u_\triangle$ *for* $\varepsilon = \triangle, \square$

(10) $u_\varepsilon . u^\triangledown . u_\triangledown = u^\triangledown . u_\triangledown$ *for* $\varepsilon = \triangle, \square$

B. *For $j = 1, \ldots, 10$, let (j') be obtained from (j) by the replacement of each u^δ $[u_\delta]$ by u_δ $[u^\delta]$, $\delta = \triangle, \triangledown, \square, \varepsilon$. Then (j') holds, too.*

C. *If (e) is obtained from the equality in (j) or (j') $(j = 1, 3, 4, 9, 10)$ by choosing $\varepsilon := \varepsilon_0 \in \{\triangle, \triangledown, \square\}$ where ε_0 does not belong to those written in (j), then for suitable (P, P_0) and u the equality (e) does not hold.*

4.2. Part B is based on the normality of the operators under consideration. Counter-examples for proving Part C can be easily presented. The theorem contains assertions which follow immediately from the other ones (by means of the simple properties mentioned in Sec. 3.3: (10) is a corollary of (9); in (3), (4), (5), (9), (10), always only one case is "essential" while the others follow from it and (2); (1) and (2) can be "reduced", too).

4.3. (We suppose the same as in Sec. 3.1; recall that symbol $\boldsymbol{p}^\varepsilon$ admits also $\boldsymbol{p}_\square$ etc. as concrete "values".) For $\boldsymbol{x} = (x_k) \in \boldsymbol{P}$ and $0 \leqq m < 1 + l(\boldsymbol{x})$, the sequence $\boldsymbol{x}^{[m]} := (x_{m+k} \mid 0 \leqq k < 1 + (l(\boldsymbol{x}) - m))$ (here $\omega_0 - m = \omega_0$) belongs to $\boldsymbol{P}$ and is said to be a *remainder of* $\boldsymbol{x}$, while $\boldsymbol{x}$ is called an *extension of* $\boldsymbol{x}^{[m]}$. We say that $\boldsymbol{A} \subseteq \boldsymbol{P}$ has the *property* I_0 $[\mathrm{I}^0]$ iff $\boldsymbol{A}$ is closed under forming remainders [extensions]. Clearly, $\boldsymbol{A}$ has I^0 $[\mathrm{I}_0]$ iff $\boldsymbol{P} - \boldsymbol{A}$ has I_0 $[\mathrm{I}^0]$. We say that an aim-mapping $\boldsymbol{p}^\varepsilon$ has *property* $K \in \{\mathrm{I}^0, \mathrm{I}_0\}$ iff $\boldsymbol{p}^\varepsilon A$ has this property for any $\boldsymbol{A} \subseteq \boldsymbol{P}$. Thus, if $\boldsymbol{p}^\varepsilon$ has I^0 $[\mathrm{I}_0]$, then $\boldsymbol{p}_\varepsilon$ has I_0 $[\mathrm{I}^0]$. In particular, $\boldsymbol{p}^\varepsilon$ $[\boldsymbol{p}_\varepsilon]$ has the property I^0 $[\mathrm{I}_0]$ for $\varepsilon = \triangle, \nabla, \square$.

Lemma 1. *Let* $\boldsymbol{p}^\varepsilon$ *either* (1) *be* $\boldsymbol{p}^\triangle$, *or* (2) *have the property* I_0. *Let* $\boldsymbol{\sigma} \in S(u)$, $x \in P$, $A \subseteq P$, $\boldsymbol{x} = (x_k) \in \mathrm{s}(x, \boldsymbol{\sigma}) \subseteq \boldsymbol{p}^\varepsilon A$, $0 \leqq m < 1 + l(\boldsymbol{x})$. *Then* $x_m \in u^\varepsilon A$ (1) *if* $\{x_0, \ldots, x_{m-1}\} \cap A = \emptyset$, (2) *always* (*respectively*).

Lemma 2. *Let* $\boldsymbol{p}^\varepsilon$ *have the property* I^0 $[\mathrm{I}_0]$. *Then* $u^\triangle \,.\, u^\varepsilon = u^\varepsilon$ $[u_\triangle \,.\, u^\varepsilon = u^\varepsilon]$.

(Cf. [3], §§ 2.1.2, 2.4.2, 2.6.2, 2 (14)–(15), 6.2, 6.5.1, 6.5.3.)

4.4. Now, assertions (2) and (3) of the theorem follow immediately by means of Sec. 4.3. In the other "essential" equalities, one inclusion follows trivially from $u_\varepsilon \subseteq \mathbf{1} \subseteq u^\varepsilon$ ($\varepsilon = \triangle, \nabla$); the other inclusion can be proved by means of Sec. 4.3 and 3.3.

As an example, let us present the idea of the proof of (8). Let $A \subseteq P$; there exists $\boldsymbol{\sigma}_1 \in S(u)$ $[\boldsymbol{\sigma}_2 \in S(u)]$ such that $\mathrm{s}(x, \boldsymbol{\sigma}_1) \subseteq \boldsymbol{p}^\square A$ $[\mathrm{s}(x, \boldsymbol{\sigma}_2) \subseteq \boldsymbol{p}^\nabla u^\square A]$ for each $x \in u^\square A$ $[x \in u^\nabla u^\square A]$. We choose $\boldsymbol{\sigma} \in S(u)$ in the following way: If $\boldsymbol{z} = (z_0, \ldots, z_l) \in \boldsymbol{Z}$, then if $z_l \notin u^\square A$, then $\boldsymbol{\sigma z} = \boldsymbol{\sigma}_2 \boldsymbol{z}$, if $z_l \in u^\square A$, then $\boldsymbol{\sigma z} = \boldsymbol{\sigma}_1(z_r, \ldots, z_l)$ where $r = \min \{k \mid 0 \leqq k \leqq l, \{z_k, \ldots, z_l\} \subseteq u^\square A\}$. Let $x \in u^\nabla u^\square A$, $\boldsymbol{x} = (x_k) \in \mathrm{s}(x, \boldsymbol{\sigma})$. Using the evident relations $A \cap P_0 \subseteq u^\square A$ and $P_0 \cup u^\square A \subseteq u^\nabla u^\square A$, and the equality $u^\nabla u^\square A = u^\triangle(P_0 \cup u^\square A)$ (Sec. 3.4), and applying both the cases – successively (2) (for $x \in u^\square A$ firstly) and (1) – of Lemma 1, we obtain that $\boldsymbol{x} \in \boldsymbol{p}_\triangle(u^\nabla u^\square A)$. Hence, $x \in u_\triangle u^\nabla u^\square A$ for any $A \subseteq P$ and $x \in u^\nabla u^\square A$. Consequently, $u^\nabla u^\square A \subseteq u_\triangle u^\nabla u^\square A \subseteq u^\nabla u^\square A$.

5. In this paper, only a minor part of the results on SN-games ([3], [5] etc.) was used; especially, the min-max results (which were proved in versions being in several ways more general, by various approaches and in connection with the investigation of the game-theoretical meaning of extreme properties of certain operators or their generalizations) have been reduced to the normality of some operators having aim-meanings, graphs of SN-games have not been used, etc. [Further, game-theoretical considerations can be performed even for $\mathscr{T}_M$; I have studied the latter possibility (cf. [6]), using some new operators (e.g., $u \to u^*$, $u^*A = u\emptyset \div u(P - A) \div uP$ ($\div$ is the symmetric difference)).]

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