Toposym 2

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THE COMPACTNESS OPERATOR
IN GENERAL TOPOLOGY

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The role of (bi)compactness has increased tremendously during the last half century. This abstract indicates a further strengthening of this notion (at the expense of the Hausdorff property, e.g.).

Let \( X \) be a set, and \( \mathcal{F} \) a family of subsets of \( X \). Let \( \varepsilon \) denote the operator, which assigns to \( \mathcal{F} \) the collection \( \varepsilon(\mathcal{F}) \), that is the family of all finite unions and arbitrary intersections of members from \( \mathcal{F} \). We do not assume that \( \varepsilon(\mathcal{F}) \) necessarily contains \( \emptyset \) and \( X \) as elements.

Such a family \( \varepsilon(\mathcal{F}) \) on a set \( X \) is called a minus-topology \((X, \varepsilon(\mathcal{F}))\) over \( X \). (It can, of course, always be extended to a topology over \( X \).)

A subset \( S \) of \( X \) is called compact relative to \( \mathcal{F} \), as usual, provided that every subfamily \( \mathcal{F}' \) of \( \mathcal{F} \), for which \( \mathcal{F}' \cup \{S\} \) has the finite intersection property, has a nonempty intersection in \( S \). So, to any \( \mathcal{F} \) corresponds a family of compact sets \( \sigma(\mathcal{F}) \) in \((X, \varepsilon(\mathcal{F}))\), where \( \sigma \) is called the compactness operator.

The elements of \( \sigma(\sigma(\mathcal{F})) = \sigma^2(\mathcal{F}) \) are called square-compact subsets of \((X, \varepsilon(\mathcal{F}))\).

A subset \( S \) of \( X \) is apparently square-compact, if every subcollection \( (\sigma(\mathcal{F}))' \) of \( \sigma(\mathcal{F}) \) for which \( (\sigma(\mathcal{F}'))' \cup \{S\} \) has the finite intersection property, has a nonempty intersection in \( S \). We call \( \sigma^2 = \sigma \) the square-compactness operator.

We have the following connections between these operators.

1. \( \varepsilon \sigma = \varepsilon \).

Observe that (1) is a reformulation of Alexander's Lemma!

2. \( \varepsilon \sigma = \sigma \varepsilon = \sigma \);
3. \( \varepsilon^2 = \varepsilon; \sigma^2 = \sigma \).

For the proof of the propositions (2) and (3) we need a lemma.

**Lemma.** Let \( C \) be a subset of \( X \) and an element of \( \sigma(\mathcal{F}) \); let \( E \) be a subset of \( X \) and an element of \( \sigma^2(\mathcal{F}) \). Then \( C \cap E \) is an element of \( \sigma^2(\mathcal{F}) \cap \sigma(\mathcal{F}) \).

**Proof.** a) Let \( \mathcal{G}' \) be a sub-collection of \( \sigma(\mathcal{F}) \) such that \( \mathcal{G}' \cup \{C \cap E\} \) has the finite intersection property. Then \( \mathcal{G}' \cup \{C\} \cup \{E\} \) has the finite intersection property.
(further written f.i.p.). But since $\mathcal{C}' \cup \{C\} \subset q(\mathcal{F})$ and $E \in q^2(\mathcal{F})$ we have $(\bigcap \mathcal{C}') \cap C \cap E \neq \emptyset$ which proves that $C \cap E \in q^3(\mathcal{F})$.

b) Choose $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}' \cup \{C \cap E\}$ has the finite intersection property. Then the collection $\mathcal{F}'' = \{F \cap C \mid F \in \mathcal{F}'\}$ has also the finite intersection property in $E$.

It is obvious that the elements of $\mathcal{F}''$ are compact relative to $\mathcal{F}$, because each element is an intersection of a subbasic closed set and a compact set. Hence $\mathcal{F}''$ is a subcollection of $q(\mathcal{F})$ with the finite intersection property in $E$ and consequently $(\bigcap \mathcal{F}'') \cap E$ is non empty. From this we obtain $(\bigcap \mathcal{F}') \cap (C \cap E) \neq \emptyset$, thus $(C \cap E) \in q^3(\mathcal{F})$.

**Proposition (2).** The collection $q^2(\mathcal{F}) = \sigma(\mathcal{F})$ is closed under finite unions and arbitrary intersections.

**Proof.** The fact that $q^2(\mathcal{F})$ is closed under finite unions is a consequence of the definition of $q^2(\mathcal{F})$. Now we will prove that $q^2(\mathcal{F})$ is closed under arbitrary intersections.

Consider a collection $\mathcal{C}' \subset q^2(\mathcal{F})$ such that $\bigcap \mathcal{C}' = E_0 \neq \emptyset$, (the case that $\bigcap \mathcal{C}' = \emptyset$ is trivial).

We must prove that every collection $\mathcal{C}'$, such that $\mathcal{C}' \cup \{E_0\}$ has the f.i.p., has a non empty intersection in $E_0$.

Pick and fix a member $E_1 \in \mathcal{C}'$ and consider the collection $\mathcal{C}'' = \{C \cap E \mid C \in \mathcal{C}'; E \in \mathcal{C}'\}$.

From the Lemma it follows that the members of $\mathcal{C}''$ are members of $q(\mathcal{F})$. By assumption $\mathcal{C}'' \cup \{E_1\}$ has the f.i.p. and hence $(\bigcap \mathcal{C}'') \cap E_1 \neq \emptyset$; but this intersection equals $(\bigcap \mathcal{C}') \cap E_0$ and this proves that $E_0 = (\bigcap \mathcal{C}') \in q^2(\mathcal{F})$.

**Proposition (3).** $q^2(\mathcal{F}) = q^4(\mathcal{F})$.

**Proof.** We first prove that $q(\mathcal{F}) \subset q^3(\mathcal{F})$. Let $C$ be an element of $q(\mathcal{F})$ and let $\mathcal{C}'$ be a subcollection of $q^2(\mathcal{F})$ such that $\mathcal{C}' \cup \{C\}$ has the f.i.p.

Pick and fix some $E_0 \in \mathcal{C}'$ and consider $\widehat{\mathcal{C}} = \{C \cap E \mid E \in \mathcal{C}'\}$.

From the Lemma it follows that each member of $\widehat{\mathcal{C}}$ is a member of $q(\mathcal{F})$ and clearly $\widehat{\mathcal{C}} \cup \{E_0\}$ has the f.i.p.

Thus $(\bigcap \mathcal{C}) \cap E_0 \neq \emptyset$; $C \cap (\bigcap \mathcal{C}') \neq \emptyset$ and hence $C$ is a member of $q^3(\mathcal{F})$, which proves that $q(\mathcal{F}) \subset q^3(\mathcal{F})$.

Similarly we can find that $q^2(\mathcal{F}) \subset q^4(\mathcal{F})$.

On the other hand $q^2(\mathcal{F})$ is defined as being the collection of compact sets relative to $q(\mathcal{F})$ and $q^4(\mathcal{F})$ as being the collection of compact sets relative to $q^3(\mathcal{F})$. From $q(\mathcal{F}) \subset q^3(\mathcal{F})$ it follows that $q^2(\mathcal{F}) \supset q^4(\mathcal{F})$.

Hence $q^2(\mathcal{F}) = q^4(\mathcal{F})$.

$\forall \sigma = \sigma$ says that for every $\mathcal{F}$ the family $q^2(\mathcal{F})$ forms a minus topology on $X$. 

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The second part of (3) tells us in particular that the $\mathcal{Q}$ operator is “of finite order” and the relations (2) and (3) determine the structure of the semigroup $\{e, \sigma\}; e$ is an identity, and $\sigma$ is an idempotent.

Let us discuss now a few special cases of importance.

I. $\mathcal{Q} = e$ holds exactly for those topological spaces in which the compact sets coincide with the closed sets. The results above become trivial.

II. $\mathcal{Q}^2 = e$. In this case $\mathcal{Q}$ and $e$ form a group of order 2 with $e$ as the identity. This case has been studied in [1]. Spaces supplied with such a minus topology are called antispaces. These are exactly those spaces in which the square-compact subsets coincide with the closed subsets. The locally compact Hausdorff spaces and the metrizable spaces are e.g. antispaces.

If $(X, \mathcal{G})$ is an antispacespace with a minus topology, then also $(X, \mathcal{Q}(\mathcal{G}))$ is an antispacespace with a minus topology. $(X, \mathcal{G})$ and $(X, \mathcal{Q}(\mathcal{G}))$ determine themselves mutually.

In particular, if $(X, \mathcal{G})$ is the real line $\mathbb{R}$, then $(X, \mathcal{Q}(\mathcal{G}))$ is an antispacespace and the corresponding topology gives us a compact non-Hausdorff $T_1$ space, denoted by $\mathcal{Q}\mathbb{R}$, and a large part of mathematics could be based onto $\mathcal{Q}\mathbb{R}$ instead of $\mathbb{R}$, since $\mathcal{Q}^2\mathbb{R} = \mathbb{R}$.

Reference