Roman Duda

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SOME NEW CONCEPTS OF DIMENSION AND THEIR GENERALIZATION

R. DUDA

Wrocław

In dimension theory there are considered several concepts of dimension, two of a few more important are: Menger-Urysohn's small inductive dimension ind (cf. [9] and [13]) and Čech's great inductive dimension Ind [4]. It is my intention to show that by their proper combination one can get $2^{\aleph_0}$ new concepts of dimension which in turn can be still further generalized. In doing so I shall follow a fashionable pattern of modern mathematics consisting in forming continuously new and new concepts and establishing continuously greater and greater generalization of what is done so far, deriving thus a proper advantage of it. And by no means I am the first to enter this path (cf. [6]).

The point I start with are the concepts of ind and Ind. Let me recall them shortly:

The *small inductive dimension* ind X of a topological space X is an integer such that:

(i) $\operatorname{ind} X = -1$ if and only if $X = \emptyset$,

(ii) $\operatorname{ind} X \leqq n$ if and only if each point of X has arbitrarily small open neighbourhoods U in X with $\operatorname{ind}(\overline{U} - U) \leqq n - 1$.

And analogously, the *great inductive dimension* Ind X of a topological space X is an integer such that:

(i) $\operatorname{Ind} X = -1$ if and only if $X = \emptyset$,

(ii) $\operatorname{Ind} X \leqq n$ if and only if each closed subset of X has arbitrarily small open neighbourhoods U in X with $\operatorname{Ind}(\overline{U} - U) \leqq n - 1$.

The two concepts coincide for metric separable spaces but, as Roy's example shows [11], they fail to coincide even for metric spaces (for spaces "between" metric separable and metric, see article [1]). Hence they are essentially distinct.

However, both are defined by induction and, as I have recently shown [3], one can assign to any sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ consisting of 0's and 1's a new notion of dimension, called γ-inductive dimension and denoted γ-ind[1]), in a way that, vaguely speaking, in a consecutive step of calculating γ-ind X for a topological space X one follows ind if $\gamma_i = 0$ or Ind if $\gamma_i = 1$.

[1]) In my papers [3]—[5] I have used notation γ-dim but γ-ind seems to be the better one.

More precisely, let Γ be the set of all sequences consisting of 0's and 1's and let X be a topological space. If $\gamma = (\gamma_1, \gamma_2, \ldots)$ is an element of Γ, then by X/γ_i we shall mean the space X itself if $\gamma_i = 0$ and the family of all closed and non-empty subsets of X if $\gamma_i = 1$.

The *γ-inductive dimension* γ-ind X of a topological space X is now an integer such that:

(i) γ-ind $X = -1$ if and only if $X = \emptyset$,

(ii) γ-ind $X \leqq n$ if and only if each element of X/γ_1 has arbitrarily small open neighbourhoods U in X with $(\gamma_2, \gamma_3, \ldots)$-ind $(\overline{U} - U) \leqq n - 1$.

Obviously, $(0, 0, \ldots)$-ind X = ind X and $(1, 1, \ldots)$-ind X = Ind X.

Hence the combination of only two concepts of dimension has yielded uncountably many different concepts of dimension. And what is still worse (from a point of view), one can prove [3] that all these concepts are essentially distinct in a sense that if α and β are different members of Γ, then α-ind $X \neq \beta$-ind X for some topological space X. I cannot help feeling a deep satisfaction at it ...

One can raise many questions related to the notion of γ-inductive dimension. Some of them have been already settled, but most, even basic ones, are still open. Let me quote a few of them.

The first question I would like to propose is the following: does there exist, for any two different members α and β of Γ, a metric space X such that α-ind $X \neq \beta$-ind X?

So far we know [3] that there exists such a finite topological T_0-space but, as Roy's example shows, one can expect a positive answer to this question. However, this conjecture still remains to be verified.

Another interesting question seems to be this: how much, for given two members α and β of Γ, can differ dimensions α-ind X and β-ind X, where X is a metric space? Or, in other words, what is the value of

$$\operatorname{dif}(\alpha, \beta) = \sup_X |\alpha\text{-ind}\, X - \beta\text{-ind}\, X|,$$

where supremum is taken over all metric spaces?

For general topological spaces, as one can show by suitable examples, this question is easily answered by saying that if $\alpha \neq \beta$, then $\operatorname{dif}(\alpha, \beta) = \infty$. And in view of Smirnov's example [12] one can expect this to be the case also for normal spaces. However, for metric spaces the question is fully open and the lack of examples other than that of Roy doesn't allow us to raise any conjecture here.

One may also try to apply the notion of γ-inductive dimension to classical dimension theory. For instance, one of the most important theorems in dimension theory is the sum theorem (cf. [10], p. 17). This, as Lokucievskiĭ's example [8] shows, does not hold even for compact Hausdorff spaces. However, one may ask the following type of questions: let X be the union of countably many closed sets,

$X = \bigcup_{j=1}^{\infty} X_j$, such that α_j-ind $X_j \leqq n$ for some $\alpha_j \in \Gamma$, where $j = 1, 2, \ldots$ Under what reasonable hypotheses either on X or on the family $\{X_j\}$ or on the family $\{\alpha_j\}$ does there exist $\gamma \in \Gamma$ such that γ-ind $X \leqq n$? Or, does there exist $k \geqq n$ such that for every $\gamma \in \Gamma$, γ-ind $X \leqq k$ holds? And so on.

A reader interested in other problems and results on γ-inductive dimension is recommended to refer to papers [4] and [5]. Let me turn now to the fulfillment of the second part of the promise, i.e. to the generalization of that notion. In doing so, I shall first generalize notions of ind and Ind following a pattern first described by Lelek [7]. Let $\mathfrak{F}$ be a topologically closed family of spaces, i.e. such that for each $F \in \mathfrak{F}$ the family $\mathfrak{F}$ contains all spaces homeomorphic to F.

The *small inductive invariant* ind $(X, \mathfrak{F})$ of a topological space X with respect to $\mathfrak{F}$ is an integer defined inductively as follows:

(i) ind $(X, \mathfrak{F}) = -1$ if and only if $X \in \mathfrak{F}$,

(ii) ind $(X, \mathfrak{F}) \leqq n$ if and only if each point of X has arbitrarily small open neighbourhoods U in X such that ind $(\overline{U} - U, \mathfrak{F}) \leqq n - 1$.

Analogously, the *great inductive invariant* Ind $(X, \mathfrak{F})$ of a topological space X with respect to $\mathfrak{F}$ is an integer defined in the following way:

(i) Ind $(X, \mathfrak{F}) = -1$ and only if $X \in \mathfrak{F}$,

(ii) Ind $(X, \mathfrak{F}) \leqq n$ if and only if each closed subset of X has arbitrarily small open neighbourhoods U in X such that Ind $(\overline{U} - U, \mathfrak{F}) \leqq n - 1$.

For instance, if $\mathfrak{F}$ is the family consisting of an empty space $\emptyset$ only, then ind $(X, \emptyset) =$ ind X and Ind $(X, \emptyset) =$ Ind X. Taking other families $\mathfrak{F}$, we receive other invariants (cf. [7]).

Now to receive the promised generalization of γ-inductive dimension we shall combine together the definitions of the small inductive invariant and the great inductive invariant in a quite analogous way to that of receiving γ-ind from ind and Ind. Namely, let Γ be the same as previously and let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be the sequence belonging to Γ. By a *γ-inductive invariant* γ-ind $(X, \mathfrak{F})$ of a topological space X with respect to $\mathfrak{F}$ we shall mean an integer such that:

(i) γ-ind $(X, \mathfrak{F}) = -1$ if and only if $X \in \mathfrak{F}$,

(ii) γ-ind $(X, \mathfrak{F}) \leqq n$ if and only if each element of X/γ_1 has arbitrarily small open neighbourhoods U in X such that $(\gamma_2, \gamma_3, \ldots)$-ind $(\overline{U} - U, \mathfrak{F}) \leqq n - 1$.

If I know little on γ-inductive dimension γ-ind, I know practically nothing on γ-inductive invariant γ-ind $(X, \mathfrak{F})$ with respect to a family $\mathfrak{F}$ not consisting of an empty set $\emptyset$ alone. All questions are here open and all paths to be followed.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES