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ON FINITE T_0-SPACES

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In 1956 the author and H. Cohen discovered a few properties of finite topological spaces with the fixed point property. One construction that came of the investigation was that of a tower space on n points. This is the topology on n points in which there is precisely one non-void open set with exactly m points for each m, $1 \leqq m \leqq n$. Thus the tower topology on the set $\{1, 2, 3\}$ has as open sets $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$ and this topology is unique up to permutation. It was used only to obtain the following existence theorem.

Theorem 1. *If n is any finite positive integer, then there is a finite space X on n points and X has the fixed point property.*

Proof. In establishing this the topology used is the tower topology and the proof is by induction. One notes that each nonvoid subset of a tower space is itself a tower space. Thus at the inductive step it is only necessary to consider maps of X onto X. But by definition of the tower topology such a map must be the identity.

Corollary 1. *The autohomeomorphism group of a finite tower space is trivial.*

Further studies of this stringent topology were undertaken by R. Hanson [1] with continuous multiplications in the space being the object of study. Finally some notes of M. McCord suggested the proper perspective for these special spaces [2].

Theorem 2. *A finite topological space X is a T_0-space if and only if X has a tower space as $1-1$ continuous image.*

Proof. The sufficiency is clear since a tower space is itself T_0. So let X be a finite T_0-space. By the T_0 property there is an open point a_1 in X; namely a minimal non-void open set in X. In $X - a_1$ the T_0 property holds and so there is a point a_2 in $X - a_1$ that is open in $X - a_1$. It follows that $\{a_1\}$ and $\{a_1, a_2\}$ are open in X. Thus by finite induction we have open sets in X that are exactly the open sets of the tower topology on a set with the same cardinality as X.

Corollary 1. *The tower topology on n points is the weakest T_0-topology on n points.*

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Corollary 2. *If X is a finite T_0-space on n points, then the points of X may be ordered $a_1, a_2, \ldots, a_n$ so that if U_i is the least open set containing a_i, $U_i - a_i \subset \bigcup_1^{i-1} U_j$, while a_1 is open.*

Corollary 3. *Each T_0 finite space contains a closed point.*

Theorem 3. *Every finite T_0 connected space is the continuous image of the closed unit interval,* $[0, 1]$.

Proof. If $|X|$ is the cardinality of X, the result is clear for $|X| = 1$. Let us suppose in the inductive hypothesis that if any point x of X is chosen that a map can be found carrying an open neighborhood of 1 onto x. We assume the result for $|X| < k$ and let $|X| = k$.

By Corollary 3 there is a point b in X and b is closed. Suppose that b separates X; that is, $X - b = A \cup B$ sep. Then $A \cup b$ and $B \cup b$ are each closed connected subsets of X. The inductive hypothesis applies to each. There is a map f_1 of $[0, \frac{1}{2}]$ onto $B \cup b$ such that $f_1(\frac{1}{2}) = b$, and a map f_2 of $[\frac{1}{2}, 1]$ onto $A \cup b$ such that $f_2(\frac{1}{2}) = b$. Since $A \cup b$, $B \cup b$ are both closed and f_1, f_2 agree on the intersection of their two domains $f : [0, 1] \to X$ given by $f \mid [0, \frac{1}{2}] = f_1$, $f \mid [\frac{1}{2}, 1] = f_2$ is a map. If we select a point c in X, then let $f^{-1}(c)$ contain the point p. Now imagine $[0, 1]$ lying in $[0, 2]$. There is a mapping g of $[0, 2]$ onto $[0, 1]$ such that $g \mid [0, 1] = i$, and $g^{-1}(p)$ contains an open neighborhood of 2. Hence fg can be used to meet the inductive hypothesis for $|X| = k$.

In case $X - b$ is connected, then by the inductive hypothesis there is a map $q : [0, \frac{1}{4}]$ onto $X - b$ such that a neighborhood of $\frac{1}{4}$ is thrown onto a point p' in a minimal neighborhood of b in X. Thus there is a map f_1 of $[0, \frac{1}{2})$ onto $X - b$ that throws $(\frac{1}{4}, \frac{1}{2})$ onto p'; just retract $[0, \frac{1}{2})$ onto $[0, \frac{1}{4}]$. Now define a function $f : [0, 1] \to X$ by $f \mid [0, \frac{1}{2}) = f_1$, $f([\frac{1}{2}, 1]) = b$. Now the minimal open set containing b in X is U_1 say. $f^{-1}(U_1) = f^{-1}(b) \cup f^{-1}(U_1 - b)$ which is open. So f is a map. Verification of the inductive hypothesis is as before.

It is now a simple matter to show that all connected finite topological spaces are continuous images of $[0, 1]$. We do this by establishing the following lemma and then applying the above theorem.

Lemma 1. *Every finite connected topological space is a continuous image of a connected finite T_0-space under a $1-1$ mapping.*

Proof. Let X be a connected finite space. If X is T_0, the result is obvious. Otherwise let a and b be points in X each of which lies in the least open set containing the other. So $X - a$ is connected.

Now our lemma is clear for $|X| = 1$. If $|X| = k$ and the result is known for $k - 1$, select a pair a and b as above. Then by the inductive hypothesis $X - a$ is a $1-1$ continuous image of a connected T_0 space Y'; $f : Y' \to X - a$. Let b' be $f^{-1}(b)$ and a'

a point not in Y'. Form the topological space $Y = Y' \cup a'$ in which a' is open while a' is to be in each basis element of Y' that contains b'. Then Y is T_0 and connected with this basis. We may now extend f to Y by defining $f(a') = a$. Then f is a map of Y onto X.

Theorem 4. *Every finite connected topological space is a continuous image of the interval.*

References

[1] *R. Hanson:* Connections between binary systems and admissible topologies. Thesis, Virginia Polytechnic Institute (1965).

[2] *M. McCord:* Finite topological spaces (Notes).