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On finite $T_0$-spaces


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In 1956 the author and H. Cohen discovered a few properties of finite topological spaces with the fixed point property. One construction that came of the investigation was that of a tower space on \( n \) points. This is the topology on \( n \) points in which there is precisely one non-void open set with exactly \( m \) points for each \( m, 1 \leq m \leq n \). Thus the tower topology on the set \( \{1, 2, 3\} \) has as open sets \( \emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\} \) and this topology is unique up to permutation. It was used only to obtain the following existence theorem.

**Theorem 1.** If \( n \) is any finite positive integer, then there is a finite space \( X \) on \( n \) points and \( X \) has the fixed point property.

**Proof.** In establishing this the topology used is the tower topology and the proof is by induction. One notes that each nonvoid subset of a tower space is itself a tower space. Thus at the inductive step it is only necessary to consider maps of \( X \) onto \( X \). But by definition of the tower topology such a map must be the identity.

**Corollary 1.** The autohomeomorphism group of a finite tower space is trivial.

Further studies of this stringent topology were undertaken by R. Hanson [1] with continuous multiplications in the space being the object of study. Finally some notes of M. McCord suggested the proper perspective for these special spaces [2].

**Theorem 2.** A finite topological space \( X \) is a \( T_0 \)-space if and only if \( X \) has a tower space as \( 1-1 \) continuous image.

**Proof.** The sufficiency is clear since a tower space is itself \( T_0 \). So let \( X \) be a finite \( T_0 \)-space. By the \( T_0 \) property there is an open point \( a_1 \) in \( X \); namely a minimal non-void open set in \( X \). In \( X - a_1 \) the \( T_0 \) property holds and so there is a point \( a_2 \) in \( X - a_1 \) that is open in \( X - a_1 \). It follows that \( \{a_1\} \) and \( \{a_1, a_2\} \) are open in \( X \). Thus by finite induction we have open sets in \( X \) that are exactly the open sets of the tower topology on a set with the same cardinality as \( X \).

**Corollary 1.** The tower topology on \( n \) points is the weakest \( T_0 \)-topology on \( n \) points.

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Corollary 2. If $X$ is a finite $T_0$-space on $n$ points, then the points of $X$ may be ordered $a_1, a_2, ..., a_n$ so that if $U_i$ is the least open set containing $a_i$, $U_i - a_i \subseteq \bigcup U_j$, while $a_1$ is open.

Corollary 3. Each $T_0$ finite space contains a closed point.

Theorem 3. Every finite $T_0$ connected space is the continuous image of the closed unit interval, $[0, 1]$.

Proof. If $|X|$ is the cardinality of $X$, the result is clear for $|X| = 1$. Let us suppose in the inductive hypothesis that if any point $x$ of $X$ is chosen that a map can be found carrying an open neighborhood of 1 onto $x$. We assume the result for $|X| < k$ and let $|X| = k$.

By Corollary 3 there is a point $b$ in $X$ and $b$ is closed. Suppose that $b$ separates $X$; that is, $X - b = A \cup B$ sep. Then $A \cup b$ and $B \cup b$ are each closed connected subsets of $X$. The inductive hypothesis applies to each. There is a map $f_1$ of $[0, \frac{1}{2}]$ onto $B \cup b$ such that $f_1(\frac{1}{2}) = b$, and a map $f_2$ of $[\frac{1}{2}, 1]$ onto $A \cup b$ such that $f_2(\frac{1}{2}) = b$. Since $A \cup b, B \cup b$ are both closed and $f_1, f_2$ agree on the intersection of their two domains $f : [0, 1] \rightarrow X$ given by $f \bigl| [0, \frac{1}{2}] = f_1, f \bigl| [\frac{1}{2}, 1] = f_2$ is a map. If we select a point $c$ in $X$, then let $f^{-1}(c)$ contain the point $p$. Now imagine $[0, 1]$ lying in $[0, 2]$. There is a mapping $g$ of $[0, 2]$ onto $[0, 1]$ such that $g \bigl| [0, 1] = i$, and $g^{-1}(p)$ contains an open neighborhood of 2. Hence $fg$ can be used to meet the inductive hypothesis for $|X| = k$.

In case $X - b$ is connected, then by the inductive hypothesis there is a map $g : [0, \frac{1}{2}]$ onto $X - b$ such that a neighborhood of $\frac{1}{2}$ is thrown onto a point $p'$ in a minimal neighborhood of $b$ in $X$. Thus there is a map $f_1$ of $[0, \frac{1}{2}]$ onto $X - b$ that throws $(\frac{1}{2}, \frac{1}{2})$ onto $p'$; just retract $[0, \frac{1}{2}]$ onto $[0, \frac{1}{3}]$. Now define a function $f : [0, 1] \rightarrow X$ by $f \bigl| [0, \frac{1}{2}] = f_1, f \bigl| [\frac{1}{2}, 1] = f_2$. Now the minimal open set containing $b$ in $X$ is $U_1$ say. $f^{-1}(U_1) = f^{-1}(b) \cup f^{-1}(U_1 - b)$ which is open. So $f$ is a map. Verification of the inductive hypothesis is as before.

It is now a simple matter to show that all connected finite topological spaces are continuous images of $[0, 1]$. We do this by establishing the following lemma and then applying the above theorem.

Lemma 1. Every finite connected topological space is a continuous image of a connected finite $T_0$-space under a $1 - 1$ mapping.

Proof. Let $X$ be a connected finite space. If $X$ is $T_0$, the result is obvious. Otherwise let $a$ and $b$ be points in $X$ each of which lies in the least open set containing the other. So $X - a$ is connected.

Now our lemma is clear for $|X| = 1$. If $|X| = k$ and the result is known for $k - 1$, select a pair $a$ and $b$ as above. Then by the inductive hypothesis $X - a$ is a $1 - 1$ continuous image of a connected $T_0$ space $Y'$; $f : Y' \rightarrow X - a$. Let $b'$ be $f^{-1}(b)$ and $a'$
a point not in $Y'$. Form the topological space $Y = Y' \cup a'$ in which $a'$ is open while $a'$ is to be in each basis element of $Y'$ that contains $b'$. Then $Y$ is $T_0$ and connected with this basis. We may now extend $f$ to $Y$ by defining $f(a') = a$. Then $f$ is a map of $Y$ onto $X$.

**Theorem 4.** Every finite connected topological space is a continuous image of the interval.

**References**
