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ON URYSOHN'S LEMMA

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In this paper we show how a well known non-tautological theorem of point-set topology can be proved in frame theory, that is in topology without points. The results are not new but were proved in the unpublished Cambridge dissertation: Dona Papert, Lattices of functions, measures and point sets, 1958.

A partially ordered set is a set \( L \) with a relation \( \leq \), such that

1) if \( a \leq b \) and \( b \leq c \) then \( a \leq c \), and
2) if \( a \leq b \) and \( b \not\leq a \) then \( a = b \).

A complete lattice is a partially ordered set such that

3) every subset \( A \) of \( L \) has a least upper bound.

The least upper bound is unique and is usually called the join of \( A \) and written \( \lor A \) or, in terms of elements, \( \lor a_\alpha \) or \( a_1 \lor a_2 \). Let \( 1 = \lor L \); then \( 1 \) is the greatest element of \( L \). Let \( 0 = \lor \emptyset \), where \( \emptyset \) is the empty set; then \( 0 \) is the least element of \( L \). The operation \( \lor \) is associative and commutative, for the join depends on the set \( A \), not on the arrangement of its elements.

If \( B \) is the set of lower bounds of \( A \), each \( a \in A \) is an upper bound of \( B \) and hence \( \lor B \leq a \). Thus \( \lor B \) is a lower bound of \( A \). This greatest lower bound of \( A \) is called the meet of \( A \) and written \( \land A \), \( \land a_\alpha \) or \( a_1 \land a_2 \). Clearly \( \land L = 0 \) and \( \land \emptyset = 1 \).

The topology \( T \) of a space \( X \), that is the set of all open sets of \( X \), is a complete lattice with the relation \( \subseteq \). For any family \( \{G_\alpha\} \) of open sets, the join \( \lor G_\alpha \) is the union \( \bigcup G_\alpha \) and the meet \( \land G_\alpha \) is the interior of the intersection \( \bigcap G_\alpha \), thus \( G_1 \land G_2 = G_1 \cap G_2 \). The elements \( 0 \) and \( 1 \) of \( T \) are \( \emptyset \) and \( X \).

A frame is a complete lattice satisfying the distributive law

4) \( a \land \lor b_\alpha = \lor(a \land b_\alpha) \).

In particular \( a \land (b \lor c) = (a \land b) \lor (a \land c) \). Also we have \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \), for \( (a \lor b) \land (a \lor c) = ((a \lor b) \land a) \lor ((a \lor b) \land c) = a \lor (a \land c) \lor (b \land c) = a \lor (b \land c) \). From 4) by commutativity we have \( (\lor a_\alpha) \land b = \lor(a_\alpha \land b) \). Applying 4) again gives \( \lor a_\alpha \land \lor b_\beta = \lor(a_\alpha \land b_\beta) = \lor \lor a_\alpha \land b_\beta \), and, by induction, \( \lor a_\alpha \land \lor b_\beta \land \ldots \land \lor c_\gamma = \lor a_\alpha \lor b_\beta \ldots \lor c_\gamma \). The topology \( T \) of a space \( X \) is clearly a frame.
If $L$ and $M$ are frames, a function $\varphi : L \to M$ is called a frame map, or simply a map, if $\varphi \bigvee a_x = \bigvee \varphi a_x$ for each family $\{a_x\}$ and $\varphi \bigwedge a_i = \bigwedge \varphi a_i$ for each finite family $\{a_i\}$. In particular, when the families are empty, we have $\varphi 0_L = 0_M$ and $\varphi 1_L = 1_M$.

Let $X_1, X_2$ be spaces with topologies $T_1, T_2$, and let $f : X_1 \to X_2$ be a continuous function. For each $G \in T_2$, $f^{-1}G \in T_1$. Also $f^{-1} \bigvee G_a = \bigvee f^{-1}G_a = \bigcup f^{-1}G_a = \bigvee f^{-1}G_a$, and, for finite families $\{G_i\}$, $f^{-1} \bigwedge G_i = f^{-1} \bigcap G_i = \bigcap f^{-1}G_i = \bigvee f^{-1}G_i$. Thus $f^{-1} : T_2 \to T_1$ is a frame map. We shall now show that all frame maps of topologies of Hausdorff spaces are obtained thus from continuous functions.

**Theorem 1.** If $X_1, X_2$ are spaces with topologies $T_1, T_2$, if $X_2$ is a Hausdorff space and if $\varphi : T_2 \to T_1$ is a frame map, there exists a unique continuous function $f : X_1 \to X_2$ such that $f^{-1} = \varphi$.

**Proof.** For any point $x \in X_1$, let $G$ be the union of all open sets $G_a$ of $X_2$ for which $x \notin \varphi G_a$. Then $\varphi G = \varphi \bigvee G_a = \bigcup \varphi G_a$, so $x \notin \varphi G$. Thus $G$ is the greatest open set of $X_2$ for which $x \notin \varphi G$.

Since $\varphi 1 = 1$, that is $\varphi X_2 = X_1$, and since $x \in X_1$, hence $G \neq X_2$. Let $y \in X_2 \setminus G$. If $z$ is any other point of the Hausdorff space $X_2$, there are disjoint open sets $U, V$ with $y \in U$, $z \in V$. Then $\varphi U \cap \varphi V = \varphi(U \cap V) = \varphi \emptyset = \emptyset$. Then $x \notin \varphi U$, $x \notin \varphi V$, so $V \subseteq G$ and $z \notin G$. Thus there is only one point $y \in X_2 \setminus G$.

For each $x \in X_1$, let $f(x)$ be the point of $X_2$ not in max $\{G : x \notin \varphi G\}$. Then for $H$ open in $X_2$, $f(x) \in H$ if and only if $x \in \varphi H$; that is $f^{-1}H = \varphi H$. Thus $f^{-1}H$ is open, so $f$ is continuous. And we have $f^{-1} = \varphi$.

If $g : X_1 \to X_2$ is another continuous function, choose $x \in X_1$ for which $g(x) = \pm f(x)$. Let $H = X_2 \setminus (g(x))$. Then $x \in f^{-1}H = \varphi H$ but $x \notin \varphi^{-1}H$. Thus $g^{-1} = \pm \varphi$. This completes the proof.

A base $B$ of a frame $L$ is a subset of $L$ such that every element of $L$ is a join of elements of $B$.

**Theorem 2.** Let $L$ and $M$ be frames, let $B$ be a base of $L$ and let $\varphi : B \to M$ be a function such that if $\{b_i\}$ is finite and $\bigwedge b_i \leq \bigvee c_a$ then $\bigwedge \varphi b_i \leq \bigvee \varphi c_a$. Then $\varphi$ extends to a frame map $\mu : L \to M$.

(When the family $\{b_i\}$ is empty, the hypothesis states that if $1 = \bigvee c_a$ then $1 = \bigvee \varphi c_a$. In particular $\bigvee \varphi c = 1$.)

**Proof.** For $h \in L$ we define $\mu h = \bigvee_{b \leq h} \varphi b$. If $b \leq c$ in $B$ then $\varphi b \leq \varphi c$. Thus for $c \in B$ we have $\mu c = \bigvee_{b \leq c} \varphi b = \varphi c$. Thus $\mu$ is an extension of $\varphi$.

If $h \leq k$ then $\mu h = \bigvee_{b \leq h} \varphi b \leq \bigvee_{b \leq k} \varphi b$; hence $\mu h \leq \mu k$. 


For a finite non-empty family \( \{ h_i \} \), \( i = 1, \ldots, n \), we have
\[
\bigwedge \mu h_i = \bigvee \phi a \land \bigvee b \land \cdots \land \bigvee c = \bigvee \bigwedge \phi a \land \cdots \land \phi c.
\]
Since \( a \land \cdots \land c \subseteq \bigwedge h_i = \bigvee b \), hence by hypothesis
\[
\phi a \land \cdots \land \phi c \subseteq \bigvee c\phi a \land \cdots \land \phi c = \mu \bigwedge h_i.
\]
Thus \( \bigwedge \mu h_i \subseteq \mu \bigwedge h_i \). But since \( \bigwedge h_i \subseteq h_i \), \( \mu \bigwedge h_i \subseteq \mu h_i \), and hence \( \mu \bigwedge h_i \subseteq \bigwedge \mu h_i \).

Therefore \( \mu \bigwedge h_i \equiv \bigwedge \mu h_i \).

In case \( \{ h_i \} \) is empty this is still true, namely \( \mu 1 \equiv 1 \), for \( \mu 1 = \bigvee b<1 \phi b = 1 \).

For any family \( \{ h_a \} \) we have \( \mu \bigvee h_a = \bigvee \phi b \). When \( b \subseteq \bigvee h_a = \bigvee c b \subseteq c \subseteq h_a \), then \( \phi b \subseteq \bigvee a \phi c = \bigvee \phi c \bigvee \mu h_a \). Hence \( \mu \bigvee h_a \subseteq \bigvee \mu h_a \). But since \( \bigvee h_a \geq h_a \), \( \mu \bigvee h_a \geq \bigvee \mu h_a \) for each \( a \) and hence \( \mu \bigvee h_a \geq \bigvee \mu h_a \). Thus in each case \( \mu \bigvee h_a = \bigvee \mu h_a \).

Thus \( \mu \) is a frame map, as was to be shown.

A frame \( L \) is called normal if, whenever \( u \lor v = 1 \), there exist \( g, h \) such that
\[
g \lor v = 1, \quad u \lor h = 1, \quad g \land h = 0.
\]

Clearly the topology of a space \( X \) is normal if and only if \( X \) is a normal space.

**Theorem 3.** If \( L \) is a normal frame and \( u \lor v = 1 \) in \( L \) there exists a frame map \( \mu : T_R \rightarrow L \), where \( T_R \) is the topology of the real line \( R \), such that \( \mu(R \setminus (0)) \subseteq u, \mu(R \setminus (1)) \subseteq v \).

Proof. Let \( Q \) be the set of rational numbers. We shall construct \( g_p, h_p \in L \) for \( p \in Q \) so that \( g_p \land h_p = 0 \) and, if \( p < q \), \( g_p \lor h_q = 1 \). When they are thus defined for \( p \) and \( q \) with \( p < q \) we have \( h_p = h_p \land 1 = h_p \land (g_p \lor h_q) = h_p \land h_q \), so \( h_p \leq h_q \), and also \( g_q = g_q \land 1 = g_q \land (g_p \lor h_q) = g_q \land g_p \) so \( g_p \geq g_q \).

The rationals between 0 and 1 are countable; call them \( r_1, r_2, \ldots \). Let \( Q_n \) consist of all the rationals \( \leq 0 \) or \( \geq 1 \) and \( r_1, r_2, \ldots, r_n \). For \( p \in Q_0 \) we define \( g_p, h_p \) as follows:
\[
g_p = 1, \quad h_p = 0 \quad \text{for} \quad p < 0; \quad g_0 = u, \quad h_0 = 0; \quad g_1 = 0, \quad h_1 = v, \quad g_p = 0, \quad h_p = 1 \quad \text{for} \quad p > 1.
\]

Suppose \( g_p, h_p \) have been defined for \( p \in Q_n \). We now define \( g_r, h_r \) for \( r = r_{n+1} \).
Take the greatest \( p \in Q_n \) with \( p < r \) and the least \( q \in Q_n \) with \( q > r \). Then \( p < q \) and \( g_p \lor h_q = 1 \). By normality there exist \( g_r, h_r \) for which \( g_r \lor h_q = 1, g_p \lor h_r = 1, g_r \land h_r = 0 \). If \( s \in Q_{n+1} \) and \( s < r \) then \( s \leq p, g_s \geq g_p \) and \( g_s \lor h_r = 1 \). If \( s > r \) then \( s \geq q, h_s \geq h_q \) and \( g_r \lor h_s = 1 \). Thus \( g_s, h_s \) with the required properties are defined for all \( s \in Q_{n+1} \). Hence by induction they can be defined for all \( s \in Q \).

Take the base \( B \subseteq T_R \) consisting of all open intervals \( (x, y) \) with \( x < y \). The function \( \phi : B \rightarrow L \) is defined by
\[
\phi(x, y) = \bigvee_{x < p < q < y} g_p \land h_q = \bigvee_{x < p} g_p \land \bigvee_{q < y} h_q.
\]
Let \((x_i, y_i), i = 1, \ldots, n\) be a non-empty finite family of intervals, and let \((x_a, y_a)\) be a family of intervals such that \(\cap (x_i, y_i) \subseteq \cup (x_a, y_a)\). Then

\[
\bigwedge \varphi(x_i, y_i) = (\bigvee_{x_1 < p_1 < q_1 < y_1} g_{p_1} \land h_{q_1}) \land \cdots \land (\bigvee_{x_n < p_n < q_n < y_n} g_{p_n} \land h_{q_n}) = \bigvee \cdots \bigvee g_{p_n} \land \bigvee h_{q_n} = \bigvee_{\max p < p < \min q} g_p \land h_q = \varphi \cap (x_i, y_i).
\]

For any rational numbers \(p, q\) such that \(\max x_i < p < q < \min y_i\), the compact interval \([p, q]\) is contained in \(\bigcup_{x_a < r < s < y_a} (r, s)\) for \(r, s\) rational. Hence \([p, q]\) is contained in some finite number of these intervals \((r, s)\), so the open interval \((p, q)\) is a finite union \(\bigcup_{j} (r_j, s_j)\) of such intervals. We may assume that no \((r_j, s_j)\) can be omitted from the union and that \((r_j, s_j)\) overlaps \((r_{j+1}, s_{j+1})\).

If \(r < t < s < u\) we have \((g_r \land h_s) \lor (g_t \land h_u) = (g_r \lor g_t) \land (g_r \lor h_u) \land (h_s \lor g_t) \land (h_s \lor h_u) = g_r \land h_u\). Hence \(g_p \land h_q = \bigvee g_{r_j} \land h_{s_j} \leq \bigvee \varphi(x_a, y_a)\).

If \(\bigcup (x_a, y_a) = R\) then \((-2, 3) \subseteq \bigcup (x_a, y_a)\) and hence \(1 = g_{-1} \land h_2 \leq \varphi(-2, 3) \leq \bigvee \varphi(x_a, y_a)\). Thus \(\bigwedge \varphi(x_i, y_i) \subseteq \bigvee \varphi(x_a, y_a)\) even when the family \((x_i, y_i)\) is empty. Therefore \(\varphi\) extends to a frame map \(\mu : T_R \to L\).

If \(x < y < 0\) then \(\varphi(x, y) = 0\). If \(0 < x < y\) then for \(x < p < q < y\) we have \(g_p \land h_q \leq g_0 = u\), and hence \(\varphi(x, y) \leq u\). Hence \(\mu(R \setminus (0)) = \bigvee_{\varphi(x, y)}^0 (x, y)\).

If \(x < y < 1\) then for \(x < p < q < y\) we have \(g_p \land h_q \leq h_1 = v\) and hence \(\varphi(x, y) \leq v\). If \(1 < x < y\) then \(\varphi(x, y) = 0\). Hence \(\mu(R \setminus (1)) = \bigvee_{\varphi(x, y)}^1 (x, y)\).

This completes the proof.

**Theorem 4 (Urysohn).** If \(E, F\) are disjoint closed sets of a normal space \(X\) there is a continuous real function \(f : X \to R\) such that \(f(x) = 0\) when \(x \in E\) and \(f(x) = 1\) when \(x \in F\).

**Proof.** Let \(U = X \setminus E, V = X \setminus F\); then \(U \cup V = X\). By Theorem 3 there is a map \(\mu : T_R \to T_X\), where \(T_X\) is the topology of \(X\), such that \(\mu(R \setminus (0)) \subseteq U, \mu(R \setminus (1)) \subseteq V\). By Theorem 1, since \(R\) is a Hausdorff space, there is a continuous function \(f : X \to R\) such that \(f^{-1} = \mu\). Since \(f^{-1}(R \setminus (0)) \subseteq U, f(E) \subseteq (0)\). And since \(f^{-1}(R \setminus (1)) \subseteq V, f(F) \subseteq (1)\). This completes the proof.

**Reference**