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Preclosed multivalued mappings


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The definition of preclosed univalued mappings has been given in our paper [4]. The purpose of the recent note is to give the definition of preclosed multivalued mappings and to show some results concerning these mappings.

Let \( f : X \rightarrow Y \) denote a multivalued mapping from a topological space \( X \) onto a topological space \( Y \), and let \( A \) be a subset of \( X \), \( B \) a subset of \( Y \).

Following V. I. Ponomarev [1] the set
\[
\bigcap_{1}^{B} = \{ x \mid x \in X, fx \cap B \neq \emptyset \}
\]
will be called the **large inverse image** of \( B \), the set
\[
\bigcap_{b}^{1}B = \{ x \mid x \in X, fx \subseteq B \}
\]
— the **small inverse image** of \( B \), the set
\[
\bigcap_{A} = \{ y \mid y \in Y, f^{-1}y \cap A \neq \emptyset \}
\]
— the **large image** of \( A \), and finally, the set
\[
\bigcap_{b}A = \{ y \mid y \in Y, f^{-1}y \subseteq A \}
\]
will be called the **small image** of \( A \).

**Definition.** A multivalued mapping \( f : X \rightarrow Y \) will be called a **preclosed mapping** if for every point \( y \) of \( Y \) and for every neighbourhood \( Of^{-1}y \) of its large inverse image \( f^{-1}y \), there exists a set \( H \) such that \( f^{-1}y \subseteq H \subseteq Of^{-1}y \) and that the large image \( fH \) of \( H \) is an open set in \( Y \).

**Remarks.** 1. The set of all interior points of a set \( M \) is called the interior of \( M \) and denoted by \( \text{Int} M \). It is easy to see that \( f : X \rightarrow Y \) is preclosed if and only if for each point \( y \in Y \) and for each neighbourhood \( Of^{-1}y \), we have \( y \in \text{Int} f(Of^{-1}y) \).

2. \( f : X \rightarrow Y \) is said to be closed (open) if \( fA \) is closed (open) for every closed (open) set \( A \subseteq X \). A moment’s consideration shows that any closed (open) mapping is a preclosed mapping.
In our paper [4] we have proved some theorems about univalued preclosed mappings. We shall mention here some interesting results (\(f: X \mapsto Y\) denotes a univalued continuous mapping):

1. Let \(f: X \mapsto Y\) be a preclosed, monotone\(^1\) mapping, and let \(A\) be a set such that \(A = f^{-1}fA\). Then if \(\dim A = 0\) we have \(\dim fA = 0\), if \(\text{ind } A = 0\) then \(\text{ind } fA = 0\), and if \(\text{Ind } A = 0\) we have \(\text{Ind } fA = 0\).

2. Let \(f: X \mapsto Y\) be a preclosed, bicom pact mapping, and let \(\omega R\) denote the weight of the space \(R\). Then we have \(\omega Y \leq \omega X\).

3. Let \(X\) and \(Y\) be Hausdorff spaces, \(aX\) — an extension of \(X\), \(cY\) — a perfect extension of \(Y\) (we use here the term due to E. G. Sklyarenko [2]). Let further \(f: X \mapsto Y\) be a preclosed mapping, which has an extension to a perfect (i.e., a closed, bicom pact, continuous) mapping \(f_{ac}: aX \mapsto cY\).

If \(f\) is a monotone\(^1\) mapping, then \(f_{ac}\) is also a monotone mapping.

We want to give some results concerning multivalued preclosed mappings. We have

**Lemma 1.** Let \(f: X \to Y\) be a multivalued mapping, and let \(G\) be an open-closed subset of \(X\). If \(f\) is a monotone\(^1\) mapping, then the large image of \(G\) coincides with the small image \(fG = f_bG\). If \(f\) is a monotone and preclosed mapping, then this image \(fG\) is also an open-closed set (of \(Y\)).

**Theorem 1.** Let \(f: X \to Y\) be a monotone and preclosed mapping. If \(Y\) is a connected space, then \(X\) is also a connected space.

Now let two infinite regular cardinal numbers \(a\) and \(b\) be given, \(a \leq b\). A set \(M\) is said to be an \([a, b]\)-compact set if from any open covering \(\gamma\) of \(M\), which has the power \(\gamma = m \leq b\), we can choose a subcovering \(\gamma'\), the power of which \(\gamma' = f < a\). The notion of \([a, b]\)-compactness has been defined by P. S. Alexandroff and P. S. Urysohn. The characterization, which we use here, is due to Yu. M. Smirnov.

A set \(M\) is said to be an \([a, \infty]\)-compact set if it is an \([a, b]\)-compact set for every \(b\).

We shall say that a set \(M\) is a **locally** \([a, b]\)-compact set if its every point has a neighbourhood \(U_x\) the closure \(\overline{U_x}\) of which is an \([a, b]\)-compact set.

\(f: X \to Y\) is said to be an \([a, b]\)-compact (\([a, \infty]\)-compact) mapping if the large inverse image \(f^{-1}y\) of every point \(y \in Y\) is an \([a, b]\)-compact (\([a, \infty]\)-compact) set.

Finally, we shall say that \(f: X \to Y\) is strongly continuous if the inverse mapping \(f^{-1}\) is both open and closed.

We have

**Theorem 2.** Let \(f\) be a strongly continuous, preclosed, \([a, \infty]\)-compact mapping from a space \(X\) onto a regular space \(Y\). Then the local \([a, b]\)-compactness will be preserved.

\(^1\) \(f: X \mapsto Y\) (\(f: X \to Y\)) is said to be monotone, if the (large) inverse image of every point \(y\) of \(Y\) is a connected set.
Theorem 3. Let \( f : X \to Y \) be a preclosed, \([a, b]\)-compact mapping and let \( Y_1 \) be an \([a, \infty]\)-compact subset of \( Y \). Then the large inverse image \( X_1 = f^{-1}Y_1 \) is an \([a, b]\)-compact set.

Remark. In the case of univalued mappings, this theorem is, in a certain sense, the generalization of a theorem, due to Yu. M. Smirnov (v. [3], theorem 5).

References