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GELFAND-NAIMARK THEOREMS FOR NON-COMMUTATIVE TOPOLOGICAL RINGS

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Representation theorems for C^*-algebras by continuous sections have been discussed by many authors [2, 4, 5, 7, 8, 10, 11, see also 3, 9]; other classes of rings like biregular rings have been described in terms of sections in a sheaf [1]. A theory that covers the sheaf concept and a similar concept allowing non-discrete stalks has not been discussed in the generality as we suggest it, although the work of Tomiyama [10] comes very close to our approach in the case of C*-algebras.

1. Uniform fields. The fundamental concept is the concept of a uniform field. A special case is described in [10] for C*-algebras.

1.1. Definition. Let E, B be topological spaces and $\pi : E \to B$ a surjective continuous function. A *section* is a continuous function $\sigma : V \to E$ from an open subset V of B into E such that $\pi \circ \sigma = 1_V$; it is a *global* section if $V = B$. The set of all global sections is denoted with $\Gamma(\pi)$. We say that σ *passes through* $x \in E$ if $\pi(x) \in \operatorname{dom} \sigma$ and $x = \sigma(\pi(x))$. The set $\{(x, y) \in E \times E : \pi(x) = \pi(y)\}$ will be denoted with $E \bigtriangledown E$. For any subsets $S, T \subset E \times E$, we let as usual $S \circ T = \{(x, y) \in E \times E$: there is a z such that $(x, z) \in S$ and $(z, y) \in T\}$ and $T^{(-1)} = \{(x, y) \in E \times E: (y, x) \in T\}$. A *field uniformity* $\mathfrak{U}$ for π is a filter on $E \bigtriangledown E$ such that each $U \in \mathfrak{U}$ contains the diagonal of $E \times E$ and that $\{U \circ U^{(-1)} : U \in \mathfrak{U}\}$ is a basis for $\mathfrak{U}$. For each section σ and each $U \in \mathfrak{U}$ we let $U(\sigma) = \{x \in E: \pi(x) \in \operatorname{dom} \sigma$ and $(\sigma(\pi(x)), x) \in U\}$. Finally, if the neighborhood filter of each point $x \in E$ has as a basis the set of sets $U(\sigma)$ with $U \in \mathfrak{U}$ and a section σ passing through x, then we call the pair $(\pi, \mathfrak{U})$ a *uniform field of uniform spaces*. The subspaces $\pi^{-1}(b)$ of E are called *stalks*: the space B is the *base space*.

Remark. If the diagonal of $E \times E$ is an element of $\mathfrak{U}$ then π is just a sheaf of sets.

There are quite a number of elementary properties which follow more or less from the definitions which we do not list here. The following, however, is relevant for the later discussion:

1.2. If, for a uniform field $(\pi, \mathfrak{U})$ of uniform spaces the set $\Gamma(\pi)$ of global sections is not empty then the set of sets $\{(\sigma, \tau) \in \Gamma(\pi) \times \Gamma(\pi) : (\sigma(b), \tau(b)) \in U$ for all $b \in B\}$, $U \in \mathfrak{U}$, is the basis of a uniform structure on $\Gamma(\pi)$.

From now on $\Gamma(\pi)$ will always denote the uniform space which is so defined. The definitions easily specialize to the case of topological groups:

1.3. Definition. Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a uniform field of uniform spaces satisfying the following additional conditions:

(a) Each stalk $\pi^{-1}(b)$ is a topological group with identity element $\varepsilon(b)$.
(b) The set of sets $U \cap (\pi^{-1}(b) \times \pi^{-1}(b))$, $U \in \mathfrak{U}$, is the left uniformity of $\pi^{-1}(b)$.
(c) The function $(x, y) \to xy^{-1} : E \nabla E \to E$ is continuous.

Then $(\pi, \mathfrak{U})$ is called a *left uniform field of topological groups*. If each stalk is in fact a topological ring then we speak of a *uniform field of topological rings* if $(\pi, \mathfrak{U})$ is a uniform field of topological groups relative to the additive groups and if the multiplication $(x, y) \to xy : E \nabla E \to E$ is continuous.

1.4. If $(\pi, \mathfrak{U})$ is a left uniform field of topological groups, then the function ε of 1.3. a is a global section and $\Gamma(\pi)$ is a topological group with identity ε under the obvious pointwise operations $(\sigma\tau)(b) = \sigma(b)\,\tau(b)$, $\sigma^{-1}(b) = \sigma(b)^{-1}$, and the uniformity of 1.2 is the left invariant uniformity relative to this topological group structure.

If $(\pi, \mathfrak{U})$ is a uniform field of topological rings then $\Gamma(\pi)$ is a ring under pointwise operations but not in general a topological ring unless additional conditions are satisfied.

The following theorem is the basis for all representation theorems in terms of continuous sections:

1.5. Theorem. *Let G be a topological group and B a set of normal subgroups. Let $E = \bigcup\{G/b : b \in B\}$ and define $\pi : E \to B$ by $\pi(x) = b$ iff $x \in G/b$. Then the following conclusions hold:*

(*i*) *There are topologies on E and B and a field uniformity $\mathfrak{U}$ for π such that $(\pi, \mathfrak{U})$ is a left uniform field of topological groups.*

(*ii*) *The topology induced on each $\pi^{-1}(b) = G/b$ is the quotient topology.*

(*iii*) *For each $g \in G$ the function $\hat{g} : B \to E$ defined by $\hat{g}(b) = bg$ (= coset of b) is a global section.*

(*iv*) *The function $g \to \hat{g}$ from G into $\Gamma(\pi)$ is a morphism of topological groups with kernel $\bigcap B$. (The canonical bijection $G/\bigcap B \to \hat{G}$ is not necessarily an isomorphism of topological groups.)*

(*v*) *The function $(x, g) \to x(g(\pi(x)))$ defines a left action of G on E which makes the pair (E, G) a transformation group with the stalks as orbits. The orbit space E/G is homeomorphic to B.*

(*vi*) *If E', B' are topological spaces having the same underlying sets as E and B such that $(\pi, \mathfrak{U})$, $\pi : E' \to B'$ is a left uniform field of topological groups satisfying (ii) and (iii), then the identity maps $E' \to E$ and $B' \to B$ are continuous.*

If G is a topological ring and B any set of ideals then $(\pi, \mathfrak{U})$ is a field of topological rings where $\mathfrak{U}$ is the field uniformity given for the additive groups by (i). *The subring $\hat{G}$ of $\Gamma(\pi)$ is a topological ring.*

The morphism $g \to \hat{g}$ is called the *Gelfand representation.*

One may note that additional structure on a topological ring usually is reflected in $\Gamma(\pi)$. Thus, e.g. if G is a C*-algebra, then $\Gamma(\pi)$ can be given the structure of a C*-algebra in a natural fashion. It should be pointed out that through Theorem 1.5 for any topological group a topology is introduced on any set B of normal subgroups. We note this in

1.6. Definition. Let B be a set of normal subgroups (resp. a set of ideals) in a topological group (resp. ring) then the topology introduced on B by 1.5 (i) is called the *weak star topology.*

2. Ideal spaces. Let A be a ring and B a set of prime ideals. Then B can be given the hull-kernel topology. To be specific we let B_s be the structure space, i.e. the space of all primitive ideals in the hull kernel topology. The advantage of the hull kernel topology is that it is often compact (e.g. if A has an identity) or locally compact (e.g if A is a C*-algebra). Its disadvantage is twofold:

Firstly, it is rarely Hausdorff, let alone completely regular which would be desirable in view of the application of Stone-Weierstrass type theorems. But secondly, by Theorem 1.5 (vi), in order that it be compatible with the natural field structure it would have to be at least as fine as the weak star topology which is frequently not the case as examples show even in the case of C*-algebras. These observations suggest modifications of the structure space. The following general lemma is a direct a consequence of the existence theorem of adjoint functors.

2.1. Let X be a topological space. Then there exists an (essentially unique) continuous map $\omega : X \to X'$ onto a completely regular space (resp. Hausdorff) space such that all continuous maps from X into a completely regular (resp. Hausdorff space) factor through φ.

2.2. Let B_s be the structure space of a ring. Let $\varphi_i : B_s \to B_i$, $i = 1, 2$ be the functions of 2.1 with B_1 completely regular and B_2 Hausdorff. For $x \in B_i$ define the ideal m_x of A by $\bigcap \varphi_i^{-1}(x)$. The set M_{cr} (resp., M_h) of all m_x, $x \in B_1$ (resp. $x \in B_2$) together with the topology that makes the function $x \to m_x$ a homeomorphism is called the *completely regularized* (resp. *separated*) *structure space* of A.

Note that M_{cr} is a surjective continuous image of M_h and that compactness of B_s implies that $M_{cr} = M_h$ is a compact Hausdorff space.

There is an example of a C*-algebra such that (despite the local compactness of B_s) the space M_{cr} is not locally compact. Tomiyama considers ideal spaces which are obtained in a similar fashion and which allow a locally compact Hausdorff topology due to a special condition [10].

2.3. Definition. Let X be any completely regular space. A *one-point compactification* of X is a compact space $X \cup \{\infty\}$, $\infty \notin X$, such that the inclusion $X \to X \cup \{\infty\}$ is bicontinuous.

One might recall that every collection of compact subsets of X defines a one point compactification by defining a basis for the neighborhood filter of ∞ by taking the complements of finite unions from the collection. Note that $X \cup \{\infty\}$ is Hausdorff iff X is locally compact and the neighborhood filter of ∞ contains all complements of compact sets.

2.4. Definition. Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a left uniform field of topological groups over a completely regular base space B. Let ε be the identity section.

Let $B \cup \{\infty\}$ be a one point compactification. Then we say that *a global section σ is zero at ∞* if for every $U \in \mathfrak{U}$ there is a set $K \subset B$ such that $(B \setminus K) \cup \{\infty\}$ is a neighborhood of ∞ in $B \cup \{\infty\}$ and that $b \notin K$ implies $(\varepsilon(b), \sigma(b)) \in U$. The set of global sections vanishing at ∞ will be denoted with $\Gamma_0(\pi)$.

2.5. $\Gamma_0(\pi)$ is a subgroup of $\Gamma(\pi)$ and a subring if $(\pi, \mathfrak{U})$ is a uniform field of rings.

For each topological ring A with the set B of its primitive ideals we have several fields arising naturally:

2.6. Definition. Let A be a topological ring, B the set of all primitive ideals, B_s the set B with the hull kernel topology, B_w the set B with the weak star topology. Let M be the completely regularized structure space, and M_w the same set with the weak star topology. Let $(\pi, \mathfrak{U})$, $\pi : E \to B_w$ and $(\pi', \mathfrak{U}')$, $\pi' : E' \to M_w$ the fields of Theorem 1.5. If the identity function $M \to M_w$ is continuous, then we let E'' be the space E' with the coarsest topology such that the identity function $E'' \to E'$ and $\pi' : E'' \to M$ are continuous. Then $(\pi'', \mathfrak{U}')$, $\pi'' : E'' \to M$ with $\pi'' = \pi'$ is also a uniform field of additive groups (which is a uniform field of topological rings if $(\pi', \mathfrak{U}')$, $\pi'' : E' \to M_w$ is one; this is the case in most applications).

3. Locally convex algebras. In order to formulate a Gelfand-Naimark type theorem for a reasonably wide class of topological algebras we need the following definition.

3.1. Definition. Let A be a topological ring. A subset S of A is called *bounded* if for each neighborhood V of 0 we have $S \subset V + \ldots + V$ (for a suitably large sum). We say that A has *bounded approximate partitions of unity relative to its structure space B_s* if there is a bounded set S such that for every $a \in A$ and every neighbourhood U of 0 and for every finite collection $C_1, \ldots, C_n$ of closed subsets of B_s with $\bigcap_1^n C_i = \emptyset$ there are elements $e_i \in I_i \cap S, I_i = \bigcap \{I : I \in C_i\}$ such that $a - (e_1 + \ldots + e_n)\, a \in U$.

We then have the following result:

3.2. Theorem. *Let A be a locally convex complete topological algebra over the reals or complexes such that multiplication and scalar multiplication are uniformly continuous on bounded sets, and such that A has a neighborhood basis of* 0 *consisting of bounded neighborhoods (this amounts to normability). Suppose the following conditions are satisfied*:

(a) *A has bounded approximate partitions of unity relative to its structure space B_s.*

(b) *For arbitrarily small open neighborhoods U of* 0 *in A the set of all $I \in B_s$ with $(a + I) \cap U = \emptyset$ is compact.*

(c) *For each neighborhood U of* 0 *there is a neighborhood V of* 0 *such that $\bigcap\{V + I : I \in B_s\} \subset U$.*

Let M be the completely regularized structure space of A (2.2).

Then $M \to M_w$ is continuous and $(\pi'', \mathfrak{U}')$, $\pi'' : E'' \to M$ (2.6) *is a uniform field of locally convex topological algebras. There is a one point compactification $M \cup \{\infty\}$ of the completely regular space M* (2.3) *such that $A \cong \Gamma_0(\pi'')$ under the Gelfand representation. If A has an identity, then M is also the separated structure space, which is compact and coincides with M_w. Then $A \cong \Gamma(\pi')$ under the Gelfand representation.*

There is, in fact, an example of a C*-algebra for which B_s is locally compact but M is not locally compact. If A does not have an identity then M_w is never Hausdorff, so that $M \neq M_w$ in that case. The proof involves Stone-Weierstrass type arguments and a number of other technical arguments.

3.3. Theorem. *Any complex* C**-algebra satisfies conditions* (a)–(c) *of* 3.2. *The isomorphisms in* 3.2 *respect the* C**-algebra structure.*

The proofs of (b) and (c) are standard whereas (a) is not quite routine; its proof is inspired by the proof of the existence of an approximate identity in a C*-algebra.

3.4. Theorem. *If A is a* C**-algebra with identity and Z its center, then the function $m \to m \cap Z$ is a homeomorphism of M onto the maximal ideal space of the center. Under the Gelfand representation Z goes onto $C(M)$.1 (where 1 is the section taking on the identity in every stalk).*

Thus 3.4 gives an algebraic characterization of the space M which was defined topologically in 2.2. Theorem 3.3 should be compared with Theorem 3.1 of Tomiyama [10]. It should be noted that the function $m \to \|\hat{a}(m)\|$, where $\hat{a}$ is the Gelfand transform of a in the field π'', is not in general continuous; the continuity of all of these functions implies the Hausdorffness of M. There is, of course, a faithful representation of A as a closed subring of continuous sections in the field $\pi : E \to B_w$, but if B_s is not Hausdorff, it will not contain all sections or even all sections vanishing at infinity. Yet the stalks in this field are primitive, whereas the structure of the stalks in the field π'' is not generally known, although every stalk is again the ring of sections over some completely regular space vanishing at infinity.

4. Weakly biregular rings. The methods outlined before can be applied in particular to discrete rings.

4.1. Definition. Let A be a ring and B_s the space of its maximal modular ideals in the hull-kernel topology. It is said to be *weakly biregular* if for an ordered pair (I_1, I_2) of maximal modular ideals there is a central idempotent e such that $e \in I_1$ and $e \notin I_2$, and if $\bigcap B_s = 0$.

Every *biregular ring* (i.e. a ring in which every principal ideal is generated by a central idempotent) is weakly biregular. The following theorem is a generalization of the representation theorem in [1] and is due to Dauns:

4.2. Theorem (Dauns). *Let A be a weakly biregular ring such that every proper ideal is contained in some maximal modular ideal. Then B_s is a totally disconnected locally compact Hausdorff space. There is a sheaf $\pi : E \to B_s$ of rings whose stalks are local rings with identity and $A \cong \Gamma_0(\pi)$ under the Gelfand representation.*

If A is biregular, then the stalks are simple rings with identity, see [1].

There are the usual converses of the representation Theorems 3.3 and 4.2 (with some qualifications).

The proofs together with a detailed discussion of a general theory of uniform fields will appear elsewhere in a joint paper with J. Dauns.

References

[1] *J. Dauns and K. H. Hofmann:* The representation of biregular rings by sheaves. Math. Z. *91* (1966), 103—123.

[2] *J. Dixmier:* Champs continus d'espaces hilbertiens et de C*-algebres II. J. Math. pures et appl. *42* (1963), 1—20.

[3] *J. Dixmier:* Les C*-algèbres et leurs représentations. Gauthier-Villars, Paris 1964.

[4] *J. Dixmier et A. Douady:* Champs continus d'espaces hilbertiens et des C*-algèbres. Bull. Soc. Math. Fr. *91* (1963), 83—106.

[5] *J. M. Fell:* The structure of algebras of operator fields. Acta Math. *106* (1961), 233—280.

[6] *J. Glimm:* A Stone Weierstrass theorem for C*-algebras. Ann. of Math. *72* (1960), 216—244.

[7] *R. Godement:* Sur la théorie des représentations unitaires. Ann. of Math. *53* (1951), 68—124.

[8] *I. Kaplansky:* The structure of certain operator algebras. Trans. Amer. Math. Soc. *70* (1951), 219—255.

[9] *M. A. Naimark:* Normed Rings. Moscow 1956.

[10] *J. Tomiyama:* Topological representation of C*-algebras. Tôhoku Math. J. *14* (1962), 187—204.

[11] *J. Tomiyama and M. Takesaki:* Applications of fibre bundles to a certain class of C*-algebras. Tôhoku Math. J. *13* (1961), 498—523.