I. Singer
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BASIC SEQUENCES AND REFLEXIVITY OF BANACH SPACES

I. SINGER

Bucureşti

R. C. JAMES has given the following characterization of reflexive Banach spaces ([2], theorem 1):

A Banach space E with a basis $\{x_n\}$ is reflexive if and only if

(a) The basis $\{x_n\}$ is *boundedly complete*, i. e. for every sequence of scalars $\{a_n\}$ such that $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < +\infty$, the series $\sum_{i=1}^{\infty} a_i x_i$ is convergent, and

(b) The basis $\{x_n\}$ is *shrinking*, i. e. $\lim_{n\to\infty} \|f\|_n = 0$ for all functionals $f \in E^*$, where $\|f\|_n$ denotes the norm of the restriction of f to the closed linear subspace of E spanned by $x_{n+1}, x_{n+2}, \dots$

Recently V. PTÁK [3] has completed the picture of the structure of reflexive Banach spaces given by this theorem, by characterizing reflexivity in terms of bounded biorthogonal systems.

Here we characterize reflexivity of a Banach space with a basis *in terms of the behaviour of its basic sequences.*

A sequence $\{z_n\} \subset E$ is called [1] a *basic sequence*, if $\{z_n\}$ is a basis of the subspace $[z_n]$ (i. e. of the closed linear subspace spanned by the sequence $\{z_n\}$). We consider the following types of basic sequences: shrinking, boundedly complete, l_+, P, P^*.

We shall say that a basic sequence $\{z_n\}$ is of type

l_+, if $\sup_n \|z_n\| < +\infty$, and if there exists a constant $\eta > 0$ such that we have, for every finite sequence $t_1, \dots, t_n \geqq 0$,

$$\|\sum_{i=1}^{n} t_i z_i\| \geqq \eta \sum_{i=1}^{n} t_i ,$$

P, if $\inf_n \|z_n\| > 0$ and $\sup_n \|\sum_{i=1}^{n} z_n\| < +\infty$,

P^*, if $\sup_n \|z_n\| < +\infty$ and $\sup_n \|\sum_{j=1}^{n} h_j\| < +\infty$, where $\{h_n\} \subset [z_n]^*$ is the sequence of functionals biorthogonal to $\{z_n\}$.

Let $\{x_n\}$ be a basis of E. Any sequence of the form

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i , \quad y_n \neq 0 \quad (n = 1, 2, \dots), \quad p_0 = 0 ,$$

is called [1] a *block basis*. We call *block subspace* of E any subspace spanned by a block basis.

In this summary we shall give only the main result (proofs and other results will appear in Studia Mathematica):

Theorem. *For a Banach space E with a basis $\{x_n\}$ the following statements are equivalent*:

(1) *E is reflexive.*

(2) *Every basis of every block subspace is shrinking.*

(3) *No basis of any block subspace is of type l_+.*

(4) *Every basis of every block subspace is boundedly complete.*

(5) *No basis of any block subspace is of type P.*

(6) *No basis of any block subspace is of type P^*.*

Corollary. *The above theorem remains valid if we replace "...basis of... block subspace" by "...basic sequence in E".*

References

[1] *C. Bessaga* and *A. Pelczyński*: On bases and unconditional convergence of series in Banach spaces. Studia math., *17* (1958), 151—164.

[2] *R. C. James*: Bases and reflexivity of Banach spaces. Ann. of Math., *52* (1950), 518—527.

[3] *V. Pták*: Biorthogonal systems and reflexivity of Banach spaces. Czechosl. Math. J., *9* (1959), p. 319—326.