Arthur Harold Stone

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NON-SEPARABLE BOREL SETS

A. H. STONE

Rochester

All spaces considered are to be metrisable and absolutely Borel. The terminology is as in C. KURATOWSKI, Topologie I. A *Borel isomorphism* f between spaces X and Y is a 1-1 mapping of X onto Y such that both f and f^{-1} take Borel sets into Borel sets. (This notion is due to G. W. MACKEY.)

The main object is to classify spaces into Borel isomorphism classes, and to characterize these classes topologically. For separable spaces it is well known that there are just two types, represented by the space of integers and by the space of all real (or irrational) numbers, respectively. It is shown that the weight of a space is invariant under Borel isomorphism, and that the number of Borel isomorphism classes of spaces of weight $\aleph_\alpha$ is exactly $|\alpha|$ if α is infinite, and at most $2^{|\alpha|+1}$ otherwise.

The proofs require the extension to general spaces of a significant part of the well-known structure theory of separable Borel sets (i. e., of weight $\aleph_0$). They are largely classical arguments, supplemented by special devices, mostly depending on paracompactness, for circumventing appeals to separability. The main steps are as follows. One first defines, and obtains the basic properties of, the analogues of the Cantor set and of the space of irrationals for spaces of weight k; when $k > \aleph_0$ these coincide in the product of $\aleph_0$ copies of a discrete set of k points; we denote this product by $B(k)$. Every space of weight k is proved to be Borel isomorphic (in fact, generalized homeomorphic) to a closed subset of $B(k)$. Next, if a continuous mapping f of a complete metric space of weight k has image of cardinal $> k$, f is shown to be a homeomorphism on some subspace of cardinal $k^{\aleph_0}$ which is homeomorphic to some $B(p)$. From this one can determine the number of Borel subsets of cardinal n of an arbitrary space of weight k, which leads to the result that weight is invariant under Borel isomorphism. One can now show that every space of weight k is Borel isomorphic to the discrete union of at most k spaces, each of the form $B(p)$ for some $p \leqq k$, and the classification mentioned above is a consequence. It is not quite complete; the unsolved problems are typified by the following: Is $B(\aleph_1)$ Borel isomorphic to the discrete union of $\aleph_1$ copies of the space $B(\aleph_0)$ of irrationals?

Characterizations are obtained only in two extreme cases: the type of a discrete set, and the type of $B(k)$ for certain values of k. Thus many problems remain. One is whether every Borel isomorphism is a generalized homeomorphism; the answer is shown to be affirmative in some cases (e. g. if one of the spaces is locally separable).

The theory can be extended to a theory of "absolutely k-analytic sets", which for $k = \aleph_0$ reduce to the analytic sets, and it is shown that the main cardinality properties carry over.

Remark. The full text appears in Rozprawy Matematyczne *28* (1962).