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DIMENSION, CATEGORY AND K(Π,n) SPACES

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This mainly deals with the problem of determining all $K(\Pi, n)$ CW-complexes of finite Lusternik-Schnirelmann category, with Π abelian. For the case $n = 1$ this problem reduces to a problem in homological algebra and for the case $n > 1$ we make use of the results of H. CARTAN on the determination of the algebras of Eilenberg-MacLane $H_*(\Pi, n; \Lambda)$ for $\Lambda = Z$ and Z_p. We deal with the cases $n = 1$ and $n > 1$ separately.

Case 1: $n = 1$. Let $Z(\Pi)$ be the group ring of Π with Π any group, not necessarily abelian. Z can be considered as a $Z(\Pi)$-module with trivial Π-operators i. e. $xm = m$ for every $x \in \Pi$ and $m \in Z$. The projective dimension of Z as a $Z(\Pi)$-module is called the cohomological dimension of Π and is denoted by $\dim \Pi$. Our first step is the following proposition on $\dim \Pi$.

Proposition. Let π be an infinite cyclic central subgroup of Π and let $\dim \Pi/\pi < \infty$. Then

$$\dim \Pi = 1 + \dim \Pi/\pi .$$

A corollary is the following:

For any infinite cyclic group π and any group Π we have

$$\dim (\Pi \times \pi) = 1 + \dim \Pi .$$

From the results of EILENBERG-GANEA [2] the problem of determining $K(\Pi, 1)$ CW-complexes of finite LUSTERNIK-SCHNIRELMANN category (Π abelian) reduces to that of determination of abelian groups Π with $\dim \Pi < \infty$, because for such groups $\dim \Pi = \text{Cat}\, K(\Pi, 1)$.

The main theorem is the following

Theorem. *If Π is abelian,* $\dim \Pi < \infty$ *if and only if Π is torsion free and of finite rank and for such groups Π (i. e. abelian groups with finite rank and no torsion).*

$$\dim \Pi = 1 + \text{rank}\, \Pi \text{ if } \Pi \text{ is not finitely generated} .$$
$$\dim \Pi = \text{rank}\, \Pi \text{ if } \Pi \text{ is finitely generated} ,$$

Case 2: $n \geqq 2$. The main theorem is the following:
Denote by Cat (Π, n) the category of any $K(\Pi, n)$ CW-complex.

Theorem. Cat $(\Pi, n) < \infty$ *if and only if n is odd and* $\Pi \approx Q^k$ *where Q denotes the additive group of the rationals and* $0 \leqq k < \infty$. *Moreover* Cat $(Q^k, 2^\mu + 1) = k$ *for any integer* $\mu \geqq 1$.

The proof utilises H. Cartan's results on $H_*(\Pi, n; Z)$ and $H_*(\Pi, n; Z_p)$ [1].

Remark. The full text appeared in the Journal of Math. and Mech., *10* (1961), 755–772.

References

[1] *H. Cartan*: Seminaire Henri Cartan 1954–55. Algèbres d'Eilenberg-MacLane et homotopie.
[2] *S. Eilenberg* and *T. Ganea*: On the Lusternik-Schnirelmann category of abstract groups. Ann. of Math., *65* (1957), 517–518.