Czesław Bessaga; A. Pełczyński

On the topological classification of complete linear metric spaces

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [87]--90.

Persistent URL: http://dml.cz/dmlcz/700953

ON THE TOPOLOGICAL CLASSIFICATION OF COMPLETE LINEAR METRIC SPACES

C. BESSAGA and A. PEŁCZYŃSKI
Warszawa

The topological classification of finite-dimensional linear spaces is quite easy. It is well-known that two finite dimensional linear spaces are homeomorphic if and only if thay have the same dimension. Moreover in this case there exists an isomorphism (linear homeomorphism) between those spaces. In the case of infinite dimensional spaces the situation is different. The topological and isomorphical (= linearly topological) clasifications are not the same. For example if $1 \leqq p' \neq p'' < +\infty$, then the spaces $l_{p'}$ and $l_{p''}$ are not isomorphic but, by a result of S. MAZUR (1930), they are homeomorphic.

One known topological invariant which distinguishes infinite-dimensional complete linear metric spaces is the density character of the space. Among the spaces of density character $\aleph$ there are $2^{\aleph}$ different isomorphical types. It is clear that the topological classification is coarser than the isomorphical one. However the hypothesis that all the spaces of the same density character are homeomorphic; is very problematic.

Now we shall consider some particular cases.

1. *Banach spaces. We know the following results*:

(1) $$l_p \sim l_2\,; \quad L_p \sim L_2 = l_2 \quad (1 \leqq p < \infty) \quad \text{(S. MAZUR [6])}.$$

(*The symbol* $X \sim Y$ *denotes that the space* X *and* Y *are homeomorphic*)

(2) $$c_0 \sim l_2 \quad \text{(M. J. KADEC [2])}.$$

(3) *Every separable infinite dimensional conjugate B-space* (*in particular every reflexive space*) *is homeomorphic to* l_2 (M. J. KADEC [3] and V. KLEE [4]).

(4) *Every B-space with density character* $\aleph$ *which contains a closed linear manifold homeomorphic to* $l_2(\aleph)$ *is homeomorphic to* $l_2(\aleph)$ (C. BESSAGA and A. PEŁCZYŃSKI [1]).

The proofs of (2) and (3) consist in the assigning to each element of the space a sequence of "coordinates" which are defined in metric terms under a convenient admissible norm. The proof of (4) makes use of some algebraic identities and properties of cartesian product of spaces. The crucial lemma is the following result of Bartle-Graves:

(5) *If* Y *is a closed linear subspace of a B-space* (*F-space*) X *then* $X \sim Y \times X/Y$ (BARTLE-GRAVES 1952). (See E. MICHAEL [7].) *In particular from* (4) *we obtain*

(4a) *Every separable B-space which contains an infinite-dimensional subspace with an unconditional basis is homeomorphic to l_2.*

We recall that the sequence (x_n) is an unconditional basis of the space X if every element x in X may be uniquely written as a sum $x = \sum_n t_n x_n$ and for every permutation (p_n), the series $\sum_n t_{p_n} x_{p_n}$ is convergent.

We deduce (4a) from (4). Let Y be a subspace of X with an unconditional basis. Then, by a result of R. C. JAMES, either Y contains a subspace isomorphic to the space c_0 or Y is isomorphic to a conjugate space. In both cases, by (3) and (2), X contains a subspace homeomorphic to l_2.

All known infinite-dimensional separable B-spaces fulfil the assumption of (4a), whence they are homeomorphic to l_2. In general case the solution of the classification problem for separable B-spaces is reduced to the positive answer to the following problem:

P. 1. Does every infinite-dimensional B-space contain an infinite-dimensional subspace with an unconditional basis?

We want to call your attention to the fact that this problem has purely linear character.

For nonseparable B-spaces the consequences of (4) are not so deep. However we can show that

(4b) $$m(\aleph) \sim l_2(2^{\aleph}) \sim l(2^{\aleph}).$$

2. Now we consider the case of locally convex metrisable complete spaces (F-spaces). It is natural to ask: Does there exist an F-space which is isomorphic to no B-space but is homeomorphic to l_2?

Let us consider the space A of all entire functions $f(z) = \sum_{n=0}^{\infty} t_n z^n$, with the topology of almost uniform convergence. Put

$$hf = (\sqrt[n]{|t_n|} \cdot t_n/|t_n|).$$

This formula defines a homeomorphic mapping from A onto the complex space c_0 (by the classical Cauchy-Hadamard formula). Hence $A \sim l_2$. In an analogical way we can prove that every nuclear Köthe space is homeomorphic to l_2. Combining this result with the theorem of Bartle-Graves and the following result of S. ROLEWICZ: [9]:

Let X be an infinite dimensional F-space. If X is isomorphic to no cartesian product $E \times s$, where E is a B-space, then X contains isomorphically a nuclear Köthe space,

we can show that all know separable infinite dimensional F-spaces except the space s are homeomorphic to l_2. Namely we obtain

(6) *Let X be an infinite-dimensional separable F-space. If either $1°$ X is isomorphic to no carthesian product $E \times s$, where E is a B-space or $2°$ $X = E \times s$ and $E \sim l_2$, then $X \sim l_2$.*

Hence we see that the classification problem for separable F-spaces is reduced to that of B-space and to the following questions:

P. 2. Is the space s homeomorphic to l_2?

About the nonseparable F-spaces we know nothing. In particular it seems to be interesting to answer the following question:

P. 3. Is the space $[l_2(\aleph)]^{\aleph_0}$ homeomorphic to $l_2(\aleph)$?[1])

For $\aleph = \aleph_0$ the answer is "yes". Morover from (6) it follows that for infinite dimensional B-space X we have $X^{\aleph_0} \sim l_2$.

3. In the case of *non-locally convex linear metric complete space* we know only a little. Mazur's result (1) can be automatically extended to the case $0 < p < 1$. However we do not know

P. 4. Is the space S of all measurable real functions on $[0, 1]$, homeomorphic to l_2?

The following problem seems to be very interesting

P. 5. Generalize the theorem of Bartle-Graves to the case of all linear metric complex spaces.

4. It seems to be interesting to introduce *other classifications* of an intermediate character between the isometrical classification and the topological one. From the analytical point of view it is natural to ask how "nice" a homeomorphism between two linear topological spaces may be. We notice that

(7) *Let X be an F-space. If there exist a B-space Y and a homeomorphism h, which uniformly maps Y onto X, then X is a B-space.*

(8) *Under the assumption that every infinite dimensional B-space is homeomorphic with its subspace of defect one, we obtain that if two spaces are homeomorphic, then they are radially homeomorphic, i. e. there is a homeomorphism h such that $\|hx\| = \|x\|$, $h(tx) = thx$.*

The Mazur homeomorphism between l_{p_1} and l_{p_2} satisfies the Hölder condition. We want to call your attention to the following unsolved questions:

P. 6. Let h be a homeomorphic mapping from a B-space X onto a B-space Y. Let us suppose that h and h^{-1} are Lipschitzian. Are then the spaces X and Y isomorphic?

P. 7. Are every two homeomorphic B-spaces uniformly homeomorphic (i. e. whether there exists such an homeomorphism h between these spaces that both h and h^{-1} are uniformly continuous).

5. Now we want to tell something about the topological equivalences of some *spaces of continuous functions*. The theorem (4a) implies

(9) *If Q is an infinite compact metric space then the space $C(Q) = R^Q$ of all real continuous functions defined on Q is homeomorphic to l_2.*

Since the space I^Q may be treated as the unit ball of $C(Q)$, then by a recent result of Klee I^Q is homeomorphic to l_2. The same is true for the spaces R_+^Q, X^Q and W^Q,

[1]) Added in proof: The answer to this problem is "yes". From this fact we can deduce that the result (4) holds true if we replace the word "B-space", by "F-space" — see [8].

where R_+ denotes the half-line, X — an arbitrary separable F-space, and W — the Hilbert cube. It seems to be very probable that

P. 8. If W is closed convex subset of an arbitrary separable B-space then W^Q is homeomorphic to l_2, for any infinite compact metric space Q.

On the other hand, if T is a separable metric space which is not an absolute retract then the space T^Q is not homeomorphic to l_2. One may ask

P. 9. Let T be a compact metric absolute retract. Is the space T^Q homeomorphic to l_2, Q being an infinite metric compact space?

We cannot answer this question even in the particular case where T is a continuum homeomorphic to the capital letter T.

References

[1] *C. Bessaga* and *A. Pełczyński*: Some remarks on homeomorphism of Banach spaces. Bull. Acad. Polon. Sci., Ser. sci. math. astr. phys. *8* (1960), 757—761.

[2] *M. J. Kadec*: On homeomorphism of certain Banach spaces (in Russian). Докл. АН СССР *92* (1953), 465—468.

[3] *M. J. Kadec*: A connection between weak and strong convergence (in Ukrainian). Zapowidi Akad. Nauk URSR, *9* (1959), 949—952.

[4] *V. Klee*: Mappings into normed linear spaces. Fund. Math. *49* (1960), 25—34.

[5] *V. Klee*: Topological equivalence of a Banach space with its unit cells. Bull. Amer. Math. Soc. *3* (1961), 286—289.

[6] *S. Mazur*: Une remarque sur l'homeomorphie des champs fonctionnels. Studia Math. *1* (1929), 83—86.

[7] *Ernest Michael*: Continuous selections I. Ann. of Mat., vol. *63* (1956), 361—382.

[8] *C. Bessaga* and *A. Pełczyński:* Some remarks on homeomorphisms of *F*-spaces. Bull. Acad. Polon. Sci., ser. sci. math. astr. phys. *10* (1962), 265—270.

[9] *C. Bessaga, A. Pełczyński* and *S. Rolewicz:* On diametral approximative dimension and linear homogenity of *F*-spaces. Ibidem *9* (1961) 677—683.