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APPLICATIONS OF TOPOLOGY TO FOUNDATIONS OF MATHEMATICS

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The subject of my talk is to give a short account of applications of Topology to Mathematical Logic. I shall restrict the discussion of the applications to the case of the classical two-valued logic and the intuitionistic logic only.

The following topological notions play an important part in metamathematical investigations:

I. 1. Totally disconnected spaces.
   2. Compactness.
   3. The Baire theorem.

II. 1. General topological spaces, in particular finite topological spaces.
   2. Algebras of open sets.
   3. Interior mappings.
   4. Strong compactness.

The first group contains notions useful in the classical logic. The second group contains notions useful in the intuitionistic logic.

Some notions mentioned above require an explanation. By a general topological space I shall understand a space satisfying the well known four axioms of Kuratowski. It is not supposed, in general, that one-point sets are closed. Consequently finite topological spaces are not discrete, in general. A space is totally disconnected if, for every pair of distinct points x, y, there exists a clopen set (i. e. a set both closed and open) A such that $x \in A$ and $y \notin A$.

By the Baire theorem I shall understand the theorem stating that no open non-void subset of a compact Hausdorff space is of the first category. Usually this theorem is formulated for complete metric spaces. The completeness does not play any essential part in logical investigations. The Baire theorem in the above formulation is due to E. ČECH [1].

It is not surprising that totally disconnected compact spaces appear in applications to the classical logic because the classical logic is closely connected with the theory of Boolean algebras. On the other hand, the M. H. STONE [27] representation theorem asserts that every Boolean algebra $\mathfrak{A}$ is isomorphic to the field of all clopen subsets of a totally disconnected compact space, i. e. there exists a totally disconnected compact space X and a mapping

$$h : \mathfrak{A} \to \mathfrak{F} = \text{the field of all clopen subsets of } X$$

such that h transforms the lattice-theoretical joins and meets in $\mathfrak{A}$ onto set-theoretical unions and intersections respectively:

$$h(a \vee b) = h(a) \cup h(b),\ h(a \wedge b) = h(a) \cap h(b), \quad \text{for } a, b \in \mathfrak{A}.$$

The Stone isomorphism h does not preserve infinite joins and meets in $\mathfrak{A}$. More precisely, if

(1) $$a = \bigvee_{t \in T} a_t \quad \text{in} \quad \mathfrak{A}$$

(i. e. if a is the smallest element which is greater than all a_t), then

$$\bigcup_{t \in T} h(a_t) \subset h(a)$$

but $\subset$ cannot be here replaced by $=$, in general. It can be easily proved that the corresponding defect set

(2) $$h(a) \setminus \bigcup_{t \in T} h(a_t)$$

is rather small, viz. it is nowhere dense in X. Similarly, if

(3) $$a = \bigwedge_{t \in T} a_t \quad \text{in} \quad \mathfrak{A}$$

(i. e. if a is the greatest element which is less than all a_t), then

$$h(a) \subset \bigcap_{t \in T} h(a_t)$$

but $\subset$ cannot be here replaced by $=$, in general. The corresponding defect set

(4) $$\bigcup_{t \in T} h(a_t) \setminus h(a)$$

is also nowhere dense.

Since the time of my talk is restricted, I shall mention only a few applications of Topology to Mathematical Logic.

Consider first a formalized mathematical theory T based on the two-valued logic. For brevity only, we shall assume that T describes only properties of some relations in a set of elements. Thus the language of the theory T contains a countable set V of signs $x, y, z, \ldots$ called individual variables and denoting arbitrary elements in the set under examination, and a finite or countable set of signs $\pi, \varrho, \ldots$ called predicates and denoting the relations under examination. Expressions like

$$\pi(x, y), \quad \varrho(x, y, z)$$

are the simplest sentences in the theory, stating that relations π, ϱ hold between elements x, y or x, y, z etc., and called elementary formulas. From elementary formulas we can form some more complicated formulas (i. e. more complicated sentences of the theory) by joining the elementary formulas by means of the connectives

$$\vee \text{ (or)}, \quad \wedge \text{ (and)}, \quad \Rightarrow \text{(if ..., then...)}, \quad - \text{ (not)}$$

and quantifiers

$$\bigvee_x \text{ (there exists an } x \text{ such that...)}, \quad \bigwedge_x \text{ (for every } x, \ldots).$$

For instance, the following expressions are formulas

$$\pi(x, y) \Rightarrow \varrho(y, z, x), \quad (\bigvee_x \varrho(x, y, z)) \Rightarrow (\pi(y, z) \wedge \pi(z, y)). \tag{5}$$

A formula which does not contain any quantifier will be called open. For instance, the first of formulas (5) is open but the second is not open.

We shall identify two formulas α, β if and only if the both implications $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ are theorems in T, i. e. they follow from the mathematical axioms of the theory T by means of the two-valued logic. After this identification, the set of all formulas becomes a Boolean algebra $\mathfrak{A}$. The element of $\mathfrak{A}$ which is determined by a formula α will be denoted by $|\alpha|$. The join, meet and the complement in $\mathfrak{A}$ are defined by the equalities

$$|\alpha| \vee |\beta| = |\alpha \vee \beta|, \quad |\alpha| \wedge |\beta| = |\alpha \wedge \beta|, \quad -|\alpha| = |-\alpha| \tag{6}$$

where, on the left side, the signs $\vee$, $\wedge$, $-$ are Boolean operations, and on the right side they are the logical connectives. Moreover it can be easily proved that, for every formula $\alpha(x)$:

$$|\bigvee_x \alpha(x)| = \bigvee_{y \in V} |\alpha(y)|, \quad |\bigwedge_x \alpha(\mu)| = \bigwedge_{y \in V} |\alpha(y)|. \tag{7}$$

These infinite joins and meets are called joins and meets corresponding to logical quantifiers. Roughly speaking, the examination of the theory T can be reduced to the examination of the Boolean algebra $\mathfrak{A}$ just defined and consequently, by the Stone representation theorem, to the examination of the Stone space of $\mathfrak{A}$.

As the first serious application of Topology to Mathematical Logic I shall mention the topological proof of the following completeness theorem of Gödel which is one of the fundamental matemathematical theorems:

Theorem I. *Every consistent theory T has a model in a countable set.*

This theorem can be formulated in the language of the theory of Boolean algebras. The equivalent Boolean formulation is as follows:

Theorem II. *For every formalized theory T, there exists an isomorphism h_0 of the corresponding Boolean algebra $\mathfrak{A}$ onto a field of subsets of a set X_0 such that h_0 transforms all infinite joins and meets corresponding to logical quantifiers onto set-theoretical unions and intersections respectively.*

Namely every point in X_0 determines, in a simple way, a model for T in a countable set.

Thus we have to find the set X_0 and the isomorphism h_0. The Stone space X for $\mathfrak{A}$ and the Stone isomorphism h are not good because h does not transform the infinite joins and meets corresponding to the logical quantifiers onto set-theoretical unions and intersections. However it is easy to correct the Stone isomorphism h to obtain the required isomorphism h_0. For every infinite join or meet (7) the corresponding defect set is nowhere dense. The number of all infinite joins and meets corresponding to logical quantifiers is $\aleph_0$. Thus the union U of all those defect sets is a set of the first category. Let

$$X_0 = X \setminus U\,, \quad h_0(a) = h(a) \setminus U \quad \text{for} \quad a \in \mathfrak{A}\,.$$

Clearly h_0 is a Boolean homomorphism. It transforms the infinite joins and meets (7) onto the corresponding unions and intersections. If a is a non-zero element of $\mathfrak{A}$, then $h_0(a) \neq 0$ by the Baire theorem. This proves that h_0 is an isomorphism and completes the topological proof of the Gödel theorem I. This proof is due to H. RASIOWA and R. SIKORSKI [9], [10] (see also L. RIEGER [16], [17]). We have seen that the Baire theorem is an essential point in this proof.

It is convenient to consider the set X_0 just defined as a topological space, the class of all sets $h_0(|\alpha|)$, where α is any open formula, being assumed as an open basis. The following theorem holds:

Theorem III. *The space X_0 is compact if and only if the theory T is open, i. e. it has a set of axioms composed of open formulas only.*

Open theories play a special part in metamathematical investigations. Roughly speaking, they are the simplest, most regular formalized theories. Theorem III yields a topological characterization of open theories. It was proved by R. SIKORSKI [19], [20], [21]. A part of it was proved, in a slight different formulation, by A. EHRENFEUCHT and A. MOSTOWSKI [2]. The case of the classical predicate calculus, i. e. of the theory with the empty set of mathematical axioms, was earlier obtained by L. RIEGER [16] (see also H. RASIOWA and R. SIKORSKI [11]).

One of fundamental theorems in Mathematical Logic is the Herbrand theorem. There exists a simple method which associates, to every formula α, a sequence

$$\alpha_1, \alpha_2, \ldots \tag{8}$$

of open formulas, called the Herbrand sequence for α. If α is given effectively, it is very easy to find the corresponding formulas (8). The Herbrand theorem states that

Theorem IV. *If the theory T is open, then, for every formula α, the following conditions are equivalent:*

(a) *α is a theorem in T,*

(b) *there exists an integer n such that the open formula $\alpha_1 \vee \ldots \vee \alpha_n$ is a theorem in T.*

The implication $(a) \Rightarrow (b)$ can be easily deduced from the compactness of X_0. If α is a theorem in T, then $h_0(|\alpha|) = X_0$. This implies, by a simple calculation, that

$$h_0(|\alpha_1|) \cup h_0(|\alpha_2|) \cup \ldots = X_0\,.$$

Since X_0 is compact and all sets $h_0(|\alpha_n|)$ are open, there exists an integer n such that

$$h_0(|\alpha_1 \vee \ldots \vee \alpha_n|) = h_0(|\alpha_1|) \cup \ldots \cup h_0(|\alpha_n|) = X_0\,.$$

Hence we infer that $\alpha_1 \vee \ldots \vee \alpha_n$ is a theorem in X_0. This is the main idea of the proof due to R. Sikorski [20], [22].

I am going to quote a few applications of Topology to the intuitionistic logic.

The investigation of classical logic leads, in a natural way, to Boolean algebras. The investigation of the intuitionistic logic leads, in a natural way, to another kind of lattices which I shall call pseudo-Boolean algebras. By definition, a lattice is said to be a pseudo-Boolean algebra if

1° it has the zero element 0,

2° for any elements a, b, the set of all x such that $a \wedge x \leqq b$ contains the greatest element.

The greatest element will be denoted by $a \Rightarrow b$. The element $a \Rightarrow 0$ will be denoted by $-a$. The operations $a \vee b$, $a \wedge b$, $a \Rightarrow b$, $-a$ are the lattice-theoretical analogues of the intuitionistic disjunction, conjunction, implication and negation respectively. The discovery of the connection between the intuitionistic logic and pseudo-Boolean algebras is due to M. H. STONE [28] and A. TARSKI [30]. Note that every pseudo-Boolean algebra has the unit element, viz. $a \Rightarrow a$ is the unit.

Pseudo-Boolean algebras are closely related to topological spaces. Let X be a topological space, and let $\mathfrak{G}(X)$ be the lattice of all open subsets of X. Then $\mathfrak{G}(X)$ is a pseudo-Boolean algebra. The lattice-theoretical join $\vee$ and meet $\wedge$ in $\mathfrak{G}(X)$ coincide with the set-theoretical union $\cup$ and intersection $\cap$. Moreover

$$A \Rightarrow B = \operatorname{int}((X \setminus A) \cup B), \quad -A = \operatorname{int}(X \setminus A)$$

for any open sets $A, B \subset X$.

Every subalgebra of $\mathfrak{G}(X)$, i. e. every subclass of $\mathfrak{G}(X)$ which is closed with respect to $\vee$, $\wedge$, $\Rightarrow$, $-$, is also a pseudo-Boolean algebra. Conversely, every pseudo-Boolean algebra is isomorphic to a subalgebra of the algebra $\mathfrak{G}(X)$ of open subset of a topological space X.

Now I can explain why interior mappings play an important part in the investigation of the intuitionistic logic. Let X_1 and X_2 be two topological spaces and let $\varphi : X_1 \to X_2$ be any mapping. We ask under what conditions the equality

$$h(A) = \varphi^{-1}(A), \quad (A \in \mathfrak{G}(X_2))$$

defines a homomorphism

$$h : \mathfrak{G}(X_2) \to \mathfrak{G}(X_1).$$

In order that h be a homomorphism it is sufficient and, under some natural additional hypotheses, also necessary that

$$\varphi^{-1}(\overline{A}) = \overline{\varphi^{-1}(A)} \quad \text{for every set} \quad A \subset X_2 \tag{9}$$

(R. SIKORSKI [23]). A. D. WALLACE [31] (see also R. SIKORSKI [26]) has proved that (9) holds if and only if φ is an interior mapping, i. e. φ is continuous and φ maps open sets onto open sets. Note that if φ is an interior mapping from X_1 onto X_2, then h is an isomorphism from $\mathfrak{G}(X_2)$ into $\mathfrak{G}(X_1)$.

Consider first the intuitionistic propositional calculus. The language of the propositional calculus contains signs $p, q, \ldots$ called propositional variables which are symbols to denote arbitrary sentences. By joining propositional variables by means of the logical connectives $\vee$, $\wedge$, $\Rightarrow$, $-$ we obtain formulas of the propositional calculus. For instance, the expression

$$(-((p \Rightarrow q) \wedge -p)) \vee q$$

is a formula in the propositional calculus.

Let X be a topological space. Any formula α in the propositional calculus can be interpreted as a topological polynomial in the space X. For this purpose it suffices to interpret propositional variables $p, q, \ldots$ as variables running through all open subsets of X (i. e. running through all elements of $\mathfrak{G}(X)$), and to interpret the symbols $\vee$, $\wedge$, $\Rightarrow$, $-$ as the signs of the lattice-theoretical operations in the pseudo-Boolean algebra $\mathfrak{G}(X)$. This polynomial will be denoted by α_X. Values of α_X are always open subsets of X, i. e. elements in $\mathfrak{G}(X)$. We shall write $\alpha_X \equiv X$ if a_X assumes only one value: the whole space X.

The following theorem shows the connection between the intuitionistic propositional calculus and Topology:

Theorem V.. *The following conditions are equivalent for every formula* α:

(i) α *is an intuitionistic propositional tautology* (*i. e. an intuitionistically true formula*),

(ii) $\alpha_X \equiv X$ *for every topological space* X.

This result is rather easy. A much deeper result is that (ii) can be here replaced by the following condition:

(ii′) $\alpha_X \equiv X$ *for every finite topological space* X.

Condition (ii′) is also equivalent to (i) and (ii). Moreover, condition (ii′) can be restricted to finite topological spaces of powers $\leqq n(\alpha)$ where $n(\alpha)$ is an integer determined, in a simple way, by the structure of the formula α. Condition (ii′) in the formulation presented here is due to J. C. C. McKinsey and A. Tarski [3], [4] but it is a topological formulation of an earlier result of Jaśkowski formulated in another language.

Conditions (ii) and (ii′) contain the quantifier "for every... space". They can be also replaced by the following equivalent condition:

(iii) $\alpha_X \equiv X$ *for a dense in itself non-void metric space* X.

This result is also due to J. C. C. McKinsey and A. Tarski [3], [4] (see also A. Tarski [1]). Since the implication (ii) $\Rightarrow$ (iii) is trivial, in order to prove the equivalence of (iii) with the remaining conditions it suffices to show that (iii) implies (ii′). This follows from the following theorem due to J. C. C. McKinsey and A. Tarski [3]:

Theorem VI. *Let* X_1 *be a non-void dense in itself metric space and let* X_2 *be a finite non-void topological space. Then there exist a dense open subset* $X_0 \subset X_1$ *and an interior mapping* φ *from* X_0 *onto* X_2.

In other words: *Then* $\mathfrak{G}(X_2)$ *is isomorphic to a subalgebra of* $\mathfrak{G}(X_0)$.

The proof of Theorem VI is rather difficult. Finite topological spaces have a complicated structure. For instance, they can contain disjoint open sets with the same boundary, etc. Analogous complicated open sets must be constructed in X_0 when φ is defined.

If X_1 is totally disconnected, we may assume that $X_0 = X_1$.

Now we shall discuss the case of the intuitionistic predicate calculus. The language of the intuitionistic predicate calculus and the definition of formulas is the same as in the case of formalized mathematical theories described earlier.

Let α be a formula in the intuitionistic predicate calculus and let X be a topological space. Similarly as in the case of propositional calculus, α can be interpreted as a topological (infinite) polynomial α_X in X whose values are open subsets in X. The exact definition is a little more complicated as in the case of the propositional calculus and therefore it is not quoted here (for details, see e. g. A. MOSTOWSKI [6] or H. RASIOWA and R. SIKORSKI [12], [13]). As previously, we shall write $\alpha_X \equiv X$ if α_X assumes only one value: the whole space X. Polynomials α_X were used for the first time by A. Mostowski [6] to the problem of verification whether a given formula is an intuitionistic tautology or not. This method of verification is based on the following theorem:

Theorem VII. *The following conditions are equivalent for every formula* α:

(i) α *is an intuitionistic predicate tautology* (*i. e. an intuitionistically true formula*);

(ii) $\alpha_X \equiv X$ *for every topological space* X.

Conditions (i), (ii) in Theorem VII are analogous to conditions (i), (ii) in Theorem V. There is no analogue of conditions (ii′) from Theorem V. The question arises whether there is an analogue of condition (iii) from Theorem V. First H. Rasiowa and R. Sikorski [12] have proved that there exists a topological T_0-space Y such that the following condition is equivalent to (i) and (ii) in Theorem VII:

(iii$_0$) $$\alpha_Y \equiv Y.$$

The problem arises whether Y can be here replaced by a metric space. This problem was solved affirmatively by R. Sikorski [23], [24], [25]. Viz. there exists a set Z of irrational numbers such that the following condition is equivalent to (i) and (ii):

(iii) $$\alpha_Z \equiv Z.$$

This result was obtained as an easy corollary of the following topological theorem due to A. ŠVARC [29] (see also V. PONOMAREV [7]):

Theorem VIII. *Every topological* T_0*-space* Y *with a countable open basis is an interior image of a set* Z *of irrational numbers.*

A topological space X_0 is said to be strongly compact if the intersection of all non-empty closed subsets is not empty. Every topological space X can be turned into a strongly compact space X_0 by adding a new point x_0 so that X is an open dense subset of X_0. By definition,

$$X_0 = X \cup (x_0).$$

As open sets in X_0 we assume all open subsets of X and the whole space X_0. In other words, we add the point x_0 to all closed subsets. Consequently (x_0) is the intersection of all non-void sets closed in X_0.

The notion of strongly compact spaces and the trivial strong compactification just mentioned do not seem to be interesting from the topological point of view. However they play an important role in the topological investigation of the intuitionistic logic. For instance, they play an important part in the proof of Theorem VI. As second application we mention a simple topological proof of the following theorem of Gödel:

Theorem IX. *If a disjunction $\alpha \vee \beta$ is an intuitionistic tautology, then either α or β is an intuitionistic tautology.*

The topological proof of Theorem IX has been given by J. C. C. McKinsey and A. Tarski [5] (in a slight different formulation; see also L. Rieger [18]) for the propositional calculus, and by H. Rasiowa and R. Sikorski [14] for the predicate calculus. H. Rasiowa and R. Sikorski [14] have also proved the following theorem using the strong compactification:

Theorem X. *If an existential formula $\bigvee_x \alpha(x)$ is an intuitionistic predicate tautology, then there exists an individual variable y such that the substitution $\alpha(y)$ is an intuitionistic predicate tautology.*

H. Rasiowa [8] has also examined analogues of Theorems IX and X for mathematical theories based on the intuitionistic logic. In all problems concerning theorems IX and X the strong compactification plays an essential part.

In my talk I have quoted only some applications of Topology to Mathematical Logic. Other applications and all details can be found in the monograph H. Rasiowa and R. Sikorski [15] to appear probably in the next year.

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