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Descriptions of Čech cohomology


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E. H. SPANIER [5] proved that, for compact spaces, a form of the Alexander-Kolmogoroff homology theory suggested by A. D. WALLACE was isomorphic to the Čech theory. W. HUREWICZ, J. DUGUNDJI and C. H. DOWKER [4] established this result for paracompact spaces, and Dowker [3] later proved isomorphism for arbitrary topological spaces. P. ALEXANDROFF has an unpublished proof of the same theorem. The purpose of this note, largely methodological, is to outline in some detail a proof of isomorphism for paracompact spaces. It is remarkable that the proof is completely elementary and non-combinatorial in character. The corresponding development for homology with coefficients in a sheaf is sketched without proof in the last section.

Čech Cohomology. We review the definition of the Čech cohomology groups of a space X with coefficient group G in order to establish the notation. Suppose \( U = \{ U(i) \}_{i \in I} \) is an (indexed) open cover of X. For each \((q + 1)\)-tuple \( s = (s_0, s_1, \ldots, s_q)\) of members of the index set \( I \), we let \( U(S) \) be the intersection \( \bigcap \{ U(s_i) : i = 0, 1, \ldots, q \} \), and we define the nerve of the cover \( U = \{ U(i) \}_{i \in I} \) to be the complex with \( q \)-dimensional simplices \( K_q(U) = \{ s : |U(s)| \text{ non-void} \} \). The \( q \)-dimensional cochain group \( C^q(U) \) is \( \{ f : f \text{ is a function on } K_q(U) \text{ to } G \} \), and the usual coboundary operator on \( C^q(U) \) to \( C^{q+1}(U) \) then defines the cohomology groups \( H^q(U) \) of the cover.

If \( V = \{ V(j) \}_{j \in J} \) is also an open cover of X then we say that \( V \) is a refinement of \( U \) iff \( V(j) \subseteq U(n_j) \) for some suitably chosen function \( n \) on \( J \) to \( I \). We call \( n \) a refining function; \( n \) induces a refining map on \( K_q(V) \) to \( K_q(U) \), which in turn induces a chain map on \( C^q(U) \) to \( C^q(V) \), and this chain map induces a refining homomorphism of \( H^q(U) \) into \( H^q(V) \). This homomorphism is independent of the particular refining function \( n \) which is chosen. The Čech cohomology group \( H^q(X) \) is defined to be the inductive limit (direct limit), under the refining homomorphisms, of the groups \( H^q(U) \) for all open covers \( U \) of \( X \).

Small Simplex Cohomology (Vietoris Type). There is a special sort of cover which is of particular interest to us. Suppose that \( N \) is an open subset of the product \( X \times X \) which contains the diagonal \( \Delta = \{ (x, x) : x \in X \} \). For each member \( x \) of \( X \) we define \( N[x] \) to be \( \{ y : (x, y) \in N \} \) and we denote by \( N^* \) the cover \( \{ N[x] : x \in X \} \). Thus the space \( X \) itself is the index set for the cover \( N^* \). It is known that, in case \( X \) is paracom-
impact, every open cover has a refinement which is of the form \( N^* \). In other words, the class of covers of the form \( N^* \) is cofinal in the class of all open covers of \( X \). The class of open neighborhoods of \( \Delta \) is directed by \( \subset \), and we notice that if \( M \) and \( N \) are open neighborhoods of the diagonal and \( M \subset N \) then the cover \( M^* \) is a refinement of the cover \( N^* \). Moreover, if \( M \subset N \) then there is a natural choice for the refining function which carries the index set \( X \) of \( M^* \) into the index set \( X \) of \( N^* \), namely the identity. The set \( K_q(M^*) \) of \( q \)-simplices of the cover \( M^* \) is in fact a subset of \( K_q(N^*) \), and the induced refining chain map of \( C^q(N^*) \) into \( C^q(M^*) \) is restriction; that is the image of \( f \in C^q(N^*) \) is \( f \mid K_q(M^*) \). It follows from these facts that the Čech group \( H^q(X) \) is isomorphic to the inductive limit under the homomorphism induced by restriction of \( H^q(N^*) \) for neighborhoods \( N \) of the diagonal in \( X \times X \).

The preceding description of Čech cohomology has a natural geometric interpretation. If we agree that a simplex \( (x_0, x_1, \ldots, x_q) \) with vertices in \( X \) is \( N \)-small if \( \bigcap \{ N[x_i] : i = 0, 1, \ldots, q \} \) is non-void, then \( K_q(N^*) \) is just the set of \( N \)-small \( q \)-simplices, so that the cohomology theory may be called a "small simplex" theory.

**Alexander-Kolmogoroff Cohomology.** We next need the fact "cohomology commutes with inductive limit". More precisely: let \( C^q(X) \) be the inductive limit, under the restriction maps, of \( C^q(N^*) \) for \( N \) a neighborhood of the diagonal in \( X \times X \). The coboundary operator on the cochain groups \( C^q(N^*) \) induces a coboundary operator on \( C^q(X) \), and thus defines a cohomology group which we may denote \( \ast H^q(X) \). It is not hard to see that \( \ast H^q(X) \) is isomorphic to the Čech group \( H^q(X) \), since each is isomorphic to a group which can be described informally as \( \{ f : f \in C^q(N^*) \text{ for some } N, \text{ and for some } M \text{ the restriction of } f \text{ to } K_q(M) \text{ is a cocycle} \} \) modulo the equivalence relation \( \{(f, g) : \text{for some neighborhood } P \text{ of the diagonal, } f \mid K_q(P) - g \mid K_q(P) \text{ is a coboundary}\} \).

We are now very close to the Alexander-Kolmogoroff cohomology theory. The set \( K_q(N^*) \) is the subset of the set \( X^{(q+1)} \) consisting of all \( (q+1) \)-tuples of points of \( X \) which are \( N \)-small. Thus \( K_q(N^*) \) is a neighborhood of the diagonal \( \Delta^{(q+1)} = \{(x_0, x_1, \ldots, x_q) : x_i = x_0 \text{ for all } i\} \), and we shall refer to \( K_q(N^*) \) as the \( N \)-neighborhood of \( \Delta^{(q+1)} \). The inductive limit \( C^q(X) \) of the groups \( C^q(N^*) \) is then, by reason of the definition of the inductive limit, the set \( \{(f, N) : f \text{ on the } N \text{-neighborhood of } \Delta^{(q+1)} \text{ to } G \} \), modulo the equivalence relation: \( (f, N) \) is equivalent to \( (g, M) \) iff for some \( P, f = g \) on the \( P \)-neighborhood of \( \Delta^{(q+1)} \). Because the space \( X \) is paracompact, the family of \( N \)-neighborhoods of \( \Delta^{(q+1)} \) is a base for the family of all neighborhoods of \( \Delta^{(q+1)} \), and consequently \( C^q(X) \) is isomorphic to the family \( F^q \) of all functions \( f \), each defined on some neighborhood of \( \Delta^{(q+1)} \) to \( G \), modulo the subset of all functions \( f \) which vanish on some neighborhood of \( \Delta^{(q+1)} \). (The isomorphism carries each equivalence class belonging to \( C^q(X) \) into the equivalence class containing it.) Finally, each equivalence class of \( F^q \) clearly contains members with domain equal to \( X^{(q+1)} \). Whence: The Čech cohomology group \( H^q(X) \) is isomorphic to the cohomology group of the chain complex with \( q \)-dimensional cochain group equal to the group of all functions on \( X^{(q+1)} \) to \( G \),
modulo the subgroup consisting of functions zero on some neighborhood of the diagonal $\Delta^{(q+1)}$. This is the Alexander-Kolmogoroff cohomology theory.

**Cohomology with Coefficients in a Sheaf.** Essentially the same reasoning as that given above yields a description of Alexander-Kolmogoroff type for the Čech cohomology group of a paracompact space $X$ with coefficients in a sheaf $\mathcal{F}$ of Abelian groups over $X$. Let $\Sigma$ be the set of all sections of $\mathcal{F}$, where sections are added pointwise, the domain of the sum of two sections being the intersection of the domains. Let $C^q$ be the set of all functions $f$ on $X^{(q+1)}$ to $\Sigma$ with the property that for each member $x$ of $X$ there is a neighborhood $U$ of $x$ such that if $s \in U^{(q+1)}$ then $U$ is a subset of domain of $f(s)$. Let $R^q$ be the equivalence relation: $R^q = \{(f, g) : \text{for } x \in X \text{ there is a neighborhood } U \text{ of } x \text{ such that } f(s) \upharpoonright U = g(s) \upharpoonright U \text{ for } s \in U^{(q+1)}\}$. The quotient group $C^q/R^q$ inherits an addition from $\Sigma$, and with the natural coboundary operator, the $q$-th cohomology group of the chain complex with $q$-th cochain group $C^q/R^q$ is isomorphic to the Čech group $H^q(X, \mathcal{F})$.

There are several variations of the above description which pretty evidently give the same cohomology groups. R. Deheuvels [2] has a related description of $H^q(X, \mathcal{F})$ in terms of objects which are “locally” functions on $X^{(q+1)}$.

Finally, the group $C^q/R^q$ has a natural representation as a family of functions on $X$. We may describe this representation in terms of the construction above as follows. For each $x \in X$ define the equivalence relation $R^q_x$ to be $\{(f, g) : \text{for some neighborhood } U \text{ of } x \text{, if } s \in U^{(q+1)} \text{ then } f(s) \upharpoonright U = g(s) \upharpoonright U\}$. Clearly $R^q = \bigcap\{R^q_x : x \in X\}$, and the natural map $F$ such that $F(f/R^q)(x) = f/R^q_x$ is therefore an isomorphism. The family of all functions $F(f/R^q)$ might well be called the group $\mathcal{A}^q$ of Alexander cochains on $X$. It evidently has the property: if $a$ and $b$ belong to $\mathcal{A}^q$ and $a(x) = b(x)$ then $a \upharpoonright U = b \upharpoonight U$ for some neighborhood $U$ of $x$. It is true, but not obvious, that a function $b$ which locally belongs to $\mathcal{A}^q$, in the sense that every point of $X$ has a neighborhood in which $b$ agrees with some member of $\mathcal{A}^q$, necessarily belongs to $\mathcal{A}^q$. In brief, $\mathcal{A}^q$ is a complete carapace in the sense of H. CARTAN [1].

**References**