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SEMI-TOPOLOGY OF TRANSFORMATION GROUPS

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In a previous paper [1] (see also [4], example 7), I have shown that, given any set M and a transformation group $\mathfrak{A}$ of M, then between the partially-ordered set (lattice, in fact) $\mathfrak{E}(M)$ of the equivalence relations of M and the partially-ordered set of the subgroups of $\mathfrak{A}$ there can be established a dual (inverse) Galois connexion (see [2]) $\mathfrak{B}(\sim)$ and $\sim(\mathfrak{B})$ with $\sim \in \mathfrak{E}(M)$, $\mathfrak{B} \subset \mathfrak{A}$, such that $\mathfrak{B} \to \mathfrak{B}(\sim(\mathfrak{B}))$ is the closure-mapping (see [3], [4]); in other words, if $\mathfrak{B}$ is a subgroup of $\mathfrak{A}$, if $\sim(\mathfrak{B})$ is the equivalence of M corresponding to (associated with) $\mathfrak{B}$, and if $\mathfrak{B}(\sim(\mathfrak{B}))$ is the subgroup of $\mathfrak{A}$ corresponding to (associated with) $\sim(\mathfrak{B})$, then:

1. $\mathfrak{B} \subset \mathfrak{B}(\sim(\mathfrak{B}))$;
2. $\mathfrak{B}_1 \subset \mathfrak{B}_2 \Rightarrow \mathfrak{B}(\sim(\mathfrak{B}_1)) \subset \mathfrak{B}(\sim(\mathfrak{B}_2))$;
3. $\mathfrak{B}(\sim(\mathfrak{B}(\sim(\mathfrak{B})))) = \mathfrak{B}(\sim(\mathfrak{B}))$.

In the present paper it will be shown that there exists a topology (in a weaker sense) of $\mathfrak{A}$, which I shall call a *semi-topology*, such that:

a) The operations of multiplication (superposition) and inversion of transformations are continuous in this semi-topology (theorem 1).

b) The closure of a subgroup $\mathfrak{B} \subset \mathfrak{A}$, in the sense of the above inverse Galois connexion coincides with the closure of $\mathfrak{B}$ with respect to the semi-topology of $\mathfrak{A}$ (theorem 2).

c) If φ is a mapping of $\mathfrak{A}$ onto a transformation group $\mathfrak{A}'$ of a set M', where φ satisfies a certain natural condition, then φ is a continuous mapping with respect to the semi-topologies of $\mathfrak{A}$ and $\mathfrak{A}'$ (theorem 3).

The present theory is not a particular case of Everett's theory [4] concerning the topology introduced in a group whose lattice of subgroups is related to another given lattice by a given Galois connexion.

1° Let M be a non-void set, $\mathfrak{A}$ a transformation group of M; let $\mathfrak{B}$ be a subgroup of $\mathfrak{A}$; then the binary relation $\sim(\mathfrak{B}) = \sim$ of M, defined by

$$a \sim b\,, \quad a, b \in M \Leftrightarrow \exists\, \tau\,, \quad \tau \in \mathfrak{B}\,, \quad \tau(a) = b$$

is an equivalence relation of M [2], which I refer to as the *equivalence associated* with $\mathfrak{B}$. Let $\sim$ be an equivalence relation of M; then the subset $\mathfrak{B}(\sim) = \mathfrak{B} \subset \mathfrak{A}$, defined by

$$\mathfrak{B} = \{\tau \mid \tau \in \mathfrak{A},\ \tau(x) \sim x, \text{ for any } x \in M\}$$

is a subgroup of $\mathfrak{A}$ [2], which I refer to as the *subgroup associated* with $\sim$. We have the following result [1]:[1])

The mappings $\sim \to \mathfrak{B}(\sim)$ *and* $\mathfrak{B} \to \sim(\mathfrak{B})$ *establish a dual (inverse) Galois connexion between* $\mathfrak{C}(M)$, *ordered by* "$\leqq$" *where* $\sim_1 \leqq \sim_2$, $\sim_1, \sim_2 \in \mathfrak{C}(M) \Leftrightarrow \Leftrightarrow (a \sim_1 b \Rightarrow a \sim_2 b)$, *and the set of all subgroups of* $\mathfrak{A}$, *ordered by inclusion; here closure is given by* $\mathfrak{B} \to \mathfrak{B}(\sim(\mathfrak{B}))$.

2° By a *semi-topological space* is meant a non-void set S of abstract elements (*points*) such that, for any $\tau \in S$, there is given a non-void family of subsets of S (the *basis of neighbourhoods* of τ) which satisfies the conditions:

a) τ belongs to all sets of its basis of neighbourhoods;

b) for $\tau_1, \tau_2 \in S$, $\tau_1 \neq \tau_2$, there exists a set belonging to the basis of neighbourhoods of τ_1, which does not contain τ_2.

In the special case where S satisfies the additional condition that, for any $\tau \in S$ and any pair of sets U_1, U_2 in the basis of neighbourhoods of τ, there exists a U_3 in the same basis, for which $U_3 \subset U_1 \cap U_2$, the space S is topological space in the usual sense [5].

In a semi-topological space S, the following terminology will be used:

1. *open subset* of S: any union of sets belonging to the various bases of neighbourhoods of points of S, or the void subset $\emptyset$;

2. *closed subset* of S: any subset $F \subset S$, whose complement $S \setminus F$ is open;

3. *neighbourhood* of a point $\tau \in S$: any open subset containing τ;

4. *closure* $\overline{M}$ of a subset $M \subset S$: the set of all $\tau \in S$ such that $U \cap M \neq \emptyset$ for any neighbourhood U of τ.

In a semi-topological space S, we have:

α) any union of open subsets is open;

β) any intersection of closed subsets is closed;

γ) $M \subset \overline{M}$, for any $M \subset S$;

δ) a subset M is closed if and only if $\overline{M} = M$;

ε) $\overline{M}$ is closed i. e. $\overline{\overline{M}} = \overline{M}$, for any $M \subset S$;

ξ) if $M_1 \subset M_2 \subset S$ then $\overline{M}_1 \subset \overline{M}_2$;

η) $\overline{M} = \bigcap_{\substack{F \text{ closed} \\ F \supset M}} F$, for any $M \subset S$;

θ) if $M = \{\tau\}$ (i. e. a single-point set), then M is closed;

ι) $\overline{\bigcup_{i=1}^{n} M_i} \supset \bigcup_{i=1}^{n} \overline{M}_i$, for any $M_1, \ldots, M_n \subset S$.

Let S, T be two semi-topological spaces. Consider the cardinal product of the sets S, T as point set; as basis of neighbourhoods of a point (σ, τ), $\sigma \in S$, $\tau \in T$, take the family of all pairs (U, V) where U and V belong to the basis of neighbourhoods of σ

[1]) See also [4] and [2] (where it is proved that the corresponding mappings are monotone).

in S and of τ in T, respectively. Thus we obtain a semi-topological space $S \times T$, which shall be called the *cartesian product* of the given spaces S, T.

Let S, S' be two semi-topological spaces; a uniform mapping $f : S \to S'$ is by definition a *continuous mapping* of S into S' if for any $\tau \in S$, and any neighbourhood U' of $f(\tau)$ in S', one can find a neighbourhood U of τ in S with $f(U) \subset U'$.

A *semi-topological group* is by definition a non-void set $\mathfrak{G}$ of abstract elements such that following conditions are fulfilled:

I. $\mathfrak{G}$ is a group with respect to a certain law of composition, denoted by "." or by juxtaposition.

II. $\mathfrak{G}$ is a semi-topological space.

III. The mappings

$$p : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \quad \text{and} \quad i : \mathfrak{G} \to \mathfrak{G}$$

defined by $p(\sigma, \tau) = \sigma \, . \, \tau$, $i(\sigma) = \sigma^{-1}$ for $\sigma, \tau \in \mathfrak{G}$ are continuous.

3° Let M be a non-void set and $\mathfrak{A}$ a transformation group of M.

As law of composition in $\mathfrak{A}$ take the superposition of transformations. Let $\tau \in \mathfrak{A}$; as basis of neighbourhoods of τ in $\mathfrak{A}$ take the family $\{\mathfrak{U}_x^\tau\}$, $x \in M$, where

$$\mathfrak{U}_x^\tau = \{\sigma \mid \sigma \in \mathfrak{A}, \ \sigma(x) = \tau(x)\} .$$

Then we have the following results:

Theorem 1. *With respect to the defined operation and basis of neighbourhoods of a point, $\mathfrak{A}$ is a semi-topological group.*

Theorem 2. *If $\mathfrak{B}$ is a subgroup of $\mathfrak{A}$, then*

$$\overline{\mathfrak{B}} = \mathfrak{B}(\sim(\mathfrak{B}))$$

(*where by $\overline{\mathfrak{B}}$ we mean the closure in the sense of the semi-topology in $\mathfrak{A}$*), *i. e. the closure of a subgroup in the Galois connexion coincides with its closure in the semi-topology of $\mathfrak{A}$.*

Theorem 3. *Let M, M' be non-void sets, $\mathfrak{A}$, $\mathfrak{A}'$ transformation groups of M, M', respectively; let f resp. φ be mappings of M onto M', resp. of $\mathfrak{A}$ onto $\mathfrak{A}'$, satisfying the condition*

$$f(\tau(x)) = (\varphi(\tau))\,(f(x)) \,, \quad \textit{for any} \quad x \in M \,, \quad \tau \in \mathfrak{A} \,;$$

then the mapping $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a group homomorphism and a continuous mapping of $\mathfrak{A}$ onto $\mathfrak{A}'$ (in the semi-topology just defined).

We mention also the following properties:

If $\mathfrak{A}$ acs regularly on M (i. e. $\tau_1(x_0) = \tau_2(x_0)$ for some $x_0 \in M$ implies $\tau_1 = \tau_2$ whenever $\tau_1, \tau_2 \in \mathfrak{A}$) *then the semi-topology of $\mathfrak{A}$ is discrete* (i. e. any $\tau \in \mathfrak{A}$ has a single-point neighbourhood $\{\tau\}$).

A subgroup $\mathfrak{B} \subset \mathfrak{T}(M)$ (where $\mathfrak{T}(M)$ is the group of all transformations of M) *is dense in* $\mathfrak{T}(M)$ (i. e. $\overline{\mathfrak{B}} = \mathfrak{T}(M)$, where the closure $\overline{\mathfrak{B}}$ is taken with respect to the semi-topology of $\mathfrak{T}(M)$) *if and only if it acts transitively on M.*

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