John R. Isbell

Mazur's theorem

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [221]--225.

Persistent URL: http://dml.cz/dmlcz/700972

MAZUR'S THEOREM

J. R. ISBELL

Seattle

Introduction. A linear topological space A is said to *satisfy Mazur's theorem* if every sequentially continuous linear functional on A is continuous. The original theorem of S. MAZUR,[1]) completed by V. PTÁK,[2]) concerns the space $C(X)$ of all continuous real-valued functions on a completely regular space X, in the topology of pointwise convergence. $C(X)$ satisfies Mazur's theorem if and only if X is functionally closed. Moreover, in any case the sequentially continuous linear functionals on $C(X)$ are precisely the finite linear combinations of evaluations at points of the Hewitt completion of X (the completion with respect to $C(X)$).

X can be removed from this description; if $C(X)^*$ is the space of all continuous linear functionals on $C(X)$, in the weak * topology, then the space of all sequentially continuous functionals is the Hewitt completion $\upsilon[C(X)^*]$. This follows from a general theorem of H. H. CORSON:[3]) if A is any linear space, in the weak topology induced by a dual space A^*, and A^* carries the weak* topology, then υA^* is the space of all linear functionals φ on A which are continuous on countable subsets. The case $A = C(X)$ is special in that every sequentially continuous functional is countably continuous.

The question remains for general A, particularly in the weak topology induced by a dual space A^*, when does A satisfy Mazur's theorem? One hopes for an answer in terms of A^*. Of course, since $(A^*)^* = A$, there is such an answer in principle. It seems unlikely that a purely topological property of A^* (as in Corson's theorem) will suffice.

The theorem of Mazur and Pták has been generalized to vector lattices of functions [4] and to spaces of differentiable functions (E. S. Thomas[4])); it will be here extended to many rings of functions. In each case, for a space (lattice or ring) of functions A on a σ-compact[5]) space X, A satisfies Mazur's theorem. However, if we drop the side conditions, there is a countable-dimensional space A of continuous functions on a compactum X (a Cantor set) which does not satisfy Mazur's theorem.

---

[1]) Unpublished, 1946. Stated in [1], p. 74.

[2]) Unpublished, about 1956.

[3]) See Theorem 1 below.

[4]) Unpublished thesis, University of Washington, 1961.

[5]) In this paper *compact* means bicompact Hausdorff. A σ-compact space is a regular space which is a countable union of compact sets.

Corson's theorem. Let A and A^* be dual real or complex linear spaces as in the Introduction. H. H. Corson has published [3] a proof of the following theorem for the special case that A is a Banach space in the weak topology. The statement in [3] is also a little overspecialized. The theorem in its present form is again due to Corson, and the changes in the proof are quite minor.

Theorem 1. *A is functionally closed if and only if every countably continuous linear functional on A^* is continuous. Moreover, in any case the countably continuous linear functionals on A^*, in the topology induced by A^*, form the Hewitt completion of A.*

The proof depends mainly on the following theorem of Bockstein [2]. Let $P = \prod S_\alpha$ be an arbitrary topological product of separable metrizable spaces; let U and V be disjoint open subsets of P. Then for some countable set I of indices α, the projection of P upon $\prod_{\alpha \in I} S_\alpha$ maps U and V into disjoint open sets.

Bockstein's theorem yields the

Corollary. *Let D be a dense subset of a product $\prod S_\alpha$ of separable metrizable spaces; let g be a continuous real-valued function on D. Then there is a countable set I of indices α such that if π denotes the projection from D into $\prod_{\alpha \in I} S_\alpha$, g is constant on each inverse set $\pi^{-1}(p)$.*

The proof is rather straightforward, using a countable basis for the real line; it is done in [3].

Proof of Theorem 1. Let p be any point of the Hewitt completion υA. Each function f in A^* has a unique continuous extension $\hat{f}$ over υA; and putting $\varphi(f) = \hat{f}(p)$, it is easy to see that we have a countably continuous linear functional φ (which is continuous only if $p \in A$).

Conversely, let φ be a countably continuous linear functional on A^*. For any countable subset K of A^*, the smallest closed linear subspace S containing K is separable, and it is not hard to see that $\varphi \mid S$ is continuous. By the Hahn-Banach theorem, then, φ coincides on K with at least one continuous linear functional, which is represented by some a in A. Let $F(K)$ be the set of all such a. As K varies, the sets $F(K)$ form a filter base for a filter $\mathfrak{F}$. It will suffice to show that $\mathfrak{F}$ is a Cauchy filter in the uniformity induced by all continuous real-valued functions on A.

Choose a Hamel basis B for A^*. The space of all linear functionals on A^* is a product of lines $P = \prod_{\alpha \in B} R_\alpha$, containing A as a dense subspace. By the corollary to Bockstein's theorem, every continuous real-valued function g on A is determined by a countable set of coordinates $K \subset B$. In the set $F(K)$, all points have the same α-th coordinates for $\alpha \in K$; thus $g[F(K)]$ is a single point, and $\mathfrak{F}$ contains small sets with respect to g. This completes the proof.

Let us note how Corson's theorem and Pták's theorem imply that the space of all sequentially continuous functionals on $C(X)$ is $\upsilon[C(X)^*]$. From Corson's theorem, this is the space of all countably continuous functionals. From Pták's theorem, every sequentially continuous functional φ is a finite combination of evaluations at points of υX. Since those evaluations are countably continuous, so is φ.

Examples. The main counterexample shows that when A is an arbitrary space of continuous functions on a compact space X, in the topology of pointwise convergence, A need not satisfy Mazur's theorem — even if X is metric, and A countable-dimensional, so that the whole dual space A^* generated by X is metric. Actually the fact that A can be taken countable-dimensional is obvious, since X compact $\Rightarrow A^*$ σ-compact $\Rightarrow$ every countably continuous linear functional on A is continuous (by Corson's theorem). Thus once we have a discontinuous, sequentially continuous functional φ on some such A, perhaps not countable-dimensional, φ must be discontinuous on some countable subset and thus on some countable-dimensional subspace.

Then I shall describe an example in which A is not countable-dimensional, but is one of the familiar Banach spaces in an unusual topology. Specifically let A be the sequence space l_1 of all real sequences $\{a_n\}$ with $\sum |a_n| < \infty$. Let l_∞ denote the Banach dual space of all bounded sequences $\{x_n\}$, coupled to A by $(a, x) = \sum a_n x_n$. In l_∞ let X denote the set of all sequences of 0's and 1's; let A^* denote the linear space generated by X. Topologized by A, X is a Cantor set and A^* is a σ-compact non-metrizable space. Clearly A^* is a proper subspace of l_∞. If A is topologized by A^*, the functionals in $l_\infty - A^*$ are not continuous; hence, by Corson's theorem, they are not countably continuous.

However, every functional in l_∞ is sequentially continuous on A. To prove this, let $x \in l_\infty$; let (a^n) be a sequence in A such that (a^n, x) does not converge to 0. It will suffice to exhibit a functional $s \in X$ such that (a^n, s) does not converge to 0. We may suppose for convenience that $(a^n, x) = 1$ for all n; and clearly we may suppose that for each m, $a_m^n \to 0$. Also, for convenience, suppose that $\|x\| = \sup |x_n| = 1$. Now since $(a^1, x) = 1$, there is a finite set S_1 of indices such that $|\sum_{m \in S_1} a_m^1| > \frac{3}{4}$. There is a larger finite set T_1 such that $\sum_{m \text{ non} \in T_1} |a_m^1| < \frac{1}{4}$. Put $n_1 = 1$. Recursively, having finite sets $S_1, \ldots, S_k$, $T_1, \ldots, T_k$, and indices $n_1, \ldots, n_k$, such that $S_i \cap T_{i-1} = \emptyset$,

$$\Big|\sum_{m \in S_i} a_m^{n_i}\Big| > \tfrac{3}{4}, \quad \text{and} \quad \sum_{m \in T_{i-1}} |a_m^{n_i}| + \sum_{m \text{ non} \in T_i} |a_m^{n_i}| < \tfrac{1}{2}, \quad \text{for} \quad i = 1, \ldots, k,$$

there is n_{k+1} so great that for all n this great, $\sum_{m \in T_k} |a_m^n| < \frac{1}{4}$. Then S_{k+1} disjoint from T_k and T_{k+1} containing $S_{k+1} \cup T_k$ can be found to complete the recursion. Finally that s such that $s_m = 1$ for m in $\cup S_k$, $s_m = 0$ otherwise, gives $|(a^n, s)| > \frac{1}{4}$ for the infinite sequence of indices $n = n_k$.

Let us glance at another example in which A^* is generated by a complete (necessarily non-compact) subset X but A^* is not even functionally closed, so that A is very far from satisfying Mazur's theorem. In the theorems of Mazur, Pták, Isbell [4] and Thomas, completeness of X always implies Mazur's theorem for A.

Let X consist of $\aleph_1$ pairs of points (p_α, q_α). Let A be the family of all real-valued functions f on X such that for some constant c, for all but finitely many α, $f(q_\alpha) =$

$= f(p_\alpha) + c$. The weak uniformity induced on X by A consists of all countable coverings; so X is complete. But clearly the functional $\varphi(f) = c$ is well-defined, countably continuous, and discontinuous.

Special results and problems. Now that we know that Mazur's theorem is closely related to completeness in many important special cases but not in general, it becomes very interesting to look at the proofs in the special cases, and to see if they apply, or why they fail, in other interesting special cases. The unpublished proofs of Mazur and Pták, together with related unpublished proofs of L. Schwartz[6]) and S. Mrowka,[7]) are apparently very specialized (for $C(X)$, except for Schwartz's result, which concerns C^∞ functions on a manifold). My proof [4] involves a rather long chain of lemmas on "patching" functions by means of lattice operations, and a notion of *support* which permits convergence arguments showing that a sequentially continuous functional is supported by a finite set and is therefore continuous.

My student E. S. THOMAS has modified my patching arguments and, using the same notion of support and the same convergent filter of sets, has proved the following: Let X be a set and A a linear family of real-valued functions on X. Suppose (1) for every $f \in A$ and every uniformly continuous C^∞ function g of one real variable, gf is in A. Further, regard X as a topological and uniform space in the weak uniformity induced by A, and suppose (2) every function locally coinciding with functions in A is in A. Then A satisfies Mazur's theorem if and only if X is complete. This result of course contains the Mazur-Pták theorem and a good theorem on differentiable manifolds. It also provides an interesting addition to the theorem of [4], applied to spaces $C(\mu X)$ of all uniformly continuous functions.

In a different direction, H. H. Corson has pointed out a device for deducing Mazur's theorem for a space A when it is known for a suitable space B. The lemma, properly formulated, has a trivial proof.

Corson's Lemma. Suppose B is a linear family of functions on a set X and A is a linear subfamily of B which is dense in some topology satisfying the first axiom of countability and finer than the topology J of pointwise convergence on X. Then with respect to J, every sequentially continuous linear functional on A has a unique sequentially continuous extension over B; and A satisfies Mazur's theorem if and only if B does.

From this lemma and the theorem of [4] one can prove:

Corollary. *Let A be a linear algebra of real-valued functions determining the topology of a Lindelöf space X, not all vanishing at any point, and such that X is a countable union of subsets on which all the functions in A are bounded. Then in the topology of pointwise convergence on X, A satisfies Mazur's theorem.*

This applies in particular to the space of real polynomials in one variable, in the topology of pointwise convergence on any infinite subset X of the real line. However,

[6]) See [5], p. 69.
[7]) See [1], p. 75.

the question is open if we take for X, say, the unit disc in the complex plane. A more interesting, equally open question: does Mazur's theorem hold for the space A of complex polynomials in the topology of pointwise convergence on the unit disc?

One concluding remark. The method of [4] may yet yield further results, even for examples (such as the complex polynomials) in which there is a proper Šilov boundary. However, for this it would be necessary to modify the notion of *support* for a functional. In [4] this is defined in terms of the usual notion of *support* for a function, which is clearly inappropriate for analytic functions. One may say, for a space A of functions on a set X, that a subset S of X *supports* a sequentially continuous linear functional φ on A when for every sequence $\{f_n\}$ in A converging to zero pointwise on S, $\varphi(f_n)$ converges to zero. For the applications in [4] and Thomas' thesis, this notion is equivalent to the other one.

Note (added in proof). I find the results of Mazur[1]) and Mrówka[7]) were published; see S. Mrówka, Studia Math *21* (1961), 1–14. My paper [4] and Thomas' thesis[4]) are combined in one paper submitted to Proc. Amer. Math. Soc.

References

[1] *П. С. Александров:* О некоторых результатах в теории топологических пространств, полученных за последние 25 лет. Успехи матем. наук *15* (1960), 25—95.

[2] *M. Bockstein*: Un théorème de séparabilité pour les produits topologiques. Fundam. Math. *35* (1948), 242–246.

[3] *H. H. Corson*: The weak topology of a Banach space. Trans. Amer. Math. Soc. (to appear).

[4] *J. R. Isbell*: Mazur's Theorem, I. Proc. Amer. Math. Soc. (to appear).

[5] *L. Schwartz*: Théorie des distributions, Tome I. Act. Sci. Ind. 1091, Paris 1950.