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A PROBLEM CONCERNING LOCALLY-A FUNCTIONS IN A COMMUTATIVE BANACH ALGEBRA A

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1. The object of this note is to point out the equivalence of a strengthened form (Problem (Δ, A)-*hol*) of the problem "*loc-A*" with a cohomological problem "Z^1", and also to present a condition sufficient to ensure that problem loc-A has a positive solution. Here A is any commutative Banach algebra A with unit.

We proceed to explain our terminology. Let $\Delta = \Delta(A)$ be the space $Hom\,(A, \mathbf{C})$ of non-zero homomorphisms of A on the complex numbers $\mathbf{C}$, with the topology induced by its embedding in A', the dual space weakly topologized. (When referring to the Stone-Jacobson topology, we will say *regularly* open, etc.)

2. The notion of holomorphic function is essential to our discussion. To introduce it, we use only the fact that A is a set of complex-valued functions on space of A'. *A simple* holomorphic function f in A' is one of the form $\varphi \circ \alpha$ where α is the mapping of A' into $\mathbf{C}^n$ induced by some n elements $a_1, \ldots, a_n$ of A_1 and φ is holomorphic in $\mathbf{C}^n$ (and thus has an open domain): $f(\xi) = \varphi(\xi(a_1), \ldots, \xi(a_n))$. We write $f \in \mathrm{hol}\,(K)$ if f is holomorphic and defined at least on a set $K \subset A'$. For such f and K we let $[f]_K$ be the (equivalence) class of all g holomorphic in A' which agree with f in a neighborhood of K. If $K = \{\xi\}$ we write $[f]_\xi$. The class (an algebra) of all $[f]_K$ we denote by $Hol\,[K]$. If $f \in hol\,(K)$ then $[f]_\xi \in Hol\,[\xi]$ for every $\xi \in K$, but the converse is not always true except when K is compact (see [2]).

When precision requires it, we say '(A', A)-holomorphic' rather than merely 'holomorphic'. Similarly, '$Hol\,[K]$' may be amplified to '(A', A)-$Hol\,[K]$'.

3. We now take note of the algebra-structure of A_1 and let $K = \Delta$. *There is an algebra-homomorphism of* $Hol\,[\Delta]$ *into* A (indeed, onto A). This is one way of formulating the operational calculus of analytic functions for A, studied by G. E. Shilov, Calderón, Waelbrock, and the writer (see [1], [3]).

We may confine our discussion to semi-simple algebras. In that case the operational calculus just says that for each $f \in hol\,(\Delta)$ there is an element $a \in A$ such that $f \mid A = \hat{a}$.

4. Now, an analogous definition of functions holomorphic on Δ can be made by replacing the A' in section 2 above at once by Δ, leading to the concept of (Δ, A)-holomorphy. An operational calculus on such a basis would be stronger than that described in section 3. It would provide a homomorphism of a certain algebra of continuous functions F on Δ, into A. The functions F allowed would be those which, in the neighborhood of each point $\xi \in \Delta$, allow a representation of the form

$$F(\eta) = \varphi(\eta(a_1), \ldots, \eta(a_n))$$

for $\eta \in U_\xi$ where U_ξ is a neighborhood of ξ in Δ. If $f \in hol\,(\Delta)$ in the sense of (A', A') holomorphy then surely $f \mid \Delta \in hol\,(\Delta)$ in the sense of (Δ, A)-holomorphy. Converesly, if each $F \in (\Delta, A)$-*hol* (Δ) were to allow an extension f into a neighborhood of Δ in A', where $f \in (A', A)$-*hol* (Δ), then the stronger form of operational calculus would be possible.

G. E. Shilov (see I) considered this type of operational calculus, but he did not show how such an extension of F could be made. Anyway, we shall call this the *problem* (Δ, A)-hol.

There is an older problem, here to be called "*loc-A*" which is contained in (Δ, A)-*hol*. However, it is best to formulate it directly. Let a function F on Δ be called *locally A* if for each $\xi \in \Delta$ there is an $a_\xi \in A$ such that $F - a_\xi = 0$ in a neighborhood of ξ in Δ. Then the *problem loc-A* is this: Is every locally -A function representable by one $\hat{a}$ on all of Δ? An algebra for which this holds is called *sectionally complete.*

5. It is easy to see that if (Δ, A)-*hol* can be solved, then every $F \in hol\,(\Delta)$ in the second sense has an extension to an $f \in hol\,(\Delta)$ in the first sense (and conversely). We now show that this problem is equivalent to a certain cohomology problem.

We continue the discussion begun in section 2.

The union of all $Hol\,[\xi]$, $\xi \in A'$ can be topologized so as to be a *sheaf* S (an analytic sheaf, in a natural sense). We need to bring in the subsheaf $Z(\Delta)$ of those germs $[f]_\xi$ for which $f = 0$ on $\Delta \cap V$, V some neighborhood of ξ. Let S_Δ be the sheaf induced by S on Δ, and let Z_Δ be the subsheaf induced on Δ by $Z(\Delta)$. Then the sequence

$$0 \to Z_\Delta \to S_\Delta \to S_\Delta/Z_\Delta \to 0$$

is exact (for details, see [2]). Hence we obtain an exact cohomology sequence

$$0 \to H^0(Z_\Delta) \xrightarrow{i} H^0(S_\Delta) \xrightarrow{q} H^0(S_\Delta/Z_\Delta) \to$$
$$\xrightarrow{r} H^1(Z_\Delta) \to H^1(S_\Delta) \to \ldots,$$

for the space Δ.

We have shown in [2] *that* $H^1(S_\Delta) = 0$. We now assert that problem (Δ, A)-*hol* is equivalent to the problem of showing that q is *onto.* To see this, we must interpret $H^0(S_\Delta)$ and $H^0(S_\Delta/Z_\Delta)$. We recall that $H^0(T)$ is isomorphic to the group of sections of T over Δ. In particular, if $\sigma \in H^0(S_\Delta)$ then each $\sigma(\xi)$ determines an $[f_\xi]_\xi \in S_\Delta(\xi \in \Delta)$ such that when η is near to ξ then $[f_\eta]_\eta = [f_\xi]_\eta$. Thus we arrive at an $f \in hol\,(\Delta)$ such that $\sigma(\xi) = [f]_\xi$. On the other hand, for $\sigma \in H^0(S_\Delta/Z_\Delta)$, and we choose $[f_\xi]_\xi$ to represent $\sigma(\xi)$, then we cannot guarantee more than $[f_\eta]_\eta - [f_\xi]_\eta \in Z_\Delta$. This means that we have an F as in section 4. Conversely, such an F determines an element of $H^0(S_\Delta/Z_\Delta)$.

Thus $H^0(S_\Delta)$ and $H^0(S_\Delta/Z_\Delta)$ represent just the classes of data appropriate to the two kinds of operational calculi. This justifies our remark about q. Now if q is *onto*, then r must be the zero map. However, the range of r is $H^1 Z_\Delta)$ because $H^1(S_\Delta) = = 0$. Thus $H^1(Z_\Delta) = 0$ if q is onto, and conversely, obviously.

This shows that problem (Δ, A)-*hol* is equivalent to *problem* Z^1: to show $H^1(Z_\Delta) = 0$.

The theorem $H^1(S_\Delta) = 0$ can be used to show that the group G of invertible elements in A, modulo its component of 1, is isomorphic to the first cohomology group $H^1(\Delta, Z)$ of Δ with integer coefficients, and is thus the same for all algebras having a space of maximal ideals homeomorphic to a given compact Hausdorff space D.

6. To our knowledge, no algebra A has been shown to be *not* sectionally complete Concerning some algebras it is not known whether they are sectionally complete (E. Bishop has found that an algebra generated by rational functions of one element with the *sup*-norm, is sectionally complete.) We should like to present a condition sufficient (but not necessary) to ensure that each locally-A function f is given by an element $a \in A$; i. e., $f = \hat{a}$. This condition we call *combinatorial semiregularity*. It asks that whenever the *interiors* of the sets $\{f = \hat{a}_1\}, \ldots, \{f = \hat{a}_n\}$ cover Δ, then there exist regularly closed sets F_i, regularly open sets W_i, such that

$$F_i \subset W_i \subset \{f = \hat{a}_i\}$$

and $\Delta = F_1 \cup \ldots \cup F_n$. *Regular* algebras are evidently combinatorially semiregular. So are those in which $\{\hat{a} = 0\}$ has interior points only if $a = 0$ ("quasi-analytic"). Quasi-analytic algebras are in a sense the opposite of regular.

Theorem. *A combinatorially semiregular algebra is sectionally complete.*

Proof. Let f be a locally-A function defined on $\Delta(A)$. By the compactnes of Δ there exist $a_1, \ldots, a_n \in A$ such that

$$\Delta = \{f = \hat{a}_1\} \cup \ldots \cup \{f = \hat{a}_n\} .$$

This insures the existence of $F_1, \ldots, F_n$, $W_1, \ldots, W_n$ as in the preceeding definition. Let E_k be the complement of W_k. Then $K = E_k \cup F_k$ is regularly closed. Let J be the closed ideal of those $x \in A$ for which $\hat{x}$ vanishes on K. Then K is the space of maximal ideals of $Q = A/J$. Since E_k, F_k are closed and open relative to K, we have (Shilov, [3]) an element in Q which is 0 on one, and 1 on the other. Let this element be the canonical image in A/J of $u_k \in A$. Then $\hat{u}_k = 1$ on F_k and $\hat{u}_k = 0$ outside W_k. Consider the element $a = u_1 a_1 + v_1 u_2 a_2 + \ldots + v_1 v_2 \ldots v_{n-1} u_n a_n$ where $v_k = 1 - u_2$. It is not hard to verify that $f = \hat{a}$.

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Bibliography

[1] *R. Arens*: The analytic-functional calculus in commutative topological algebras. Pacific Journal of Mathematics *11* (1961), 405—429.

[2] *R. Arens*: The problem of locally-*A* functions in a commutative Banach algebra *A*, Transactions of the American Mathematical Society, to appear in 1962.

[3] *G. E. Shilov*: On the decomposition of a commutative normed ring into a direct sum of ideals, Матем. сб., *32* (1953, in Russian), 353—364.

[4] *G. E. Shilov*: Analytic functions in a normed ring. Успехи мат. наук, *15* (1960), no. 3 (93) (in Russian), 181—183.