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CONCERNING THE WEIGHT OF TOPOLOGICAL SPACES

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1. The general addition formula. Let the space be represented as a sum of its subspaces X_α:

$$X = \bigcup_\alpha X_\alpha .$$

Let wX, wX_α be the weights of the corresponding spaces. Under what conditions the formula

(1) $$wX = \sum_\alpha wX_\alpha$$

does hold?

In their "Mémoire sur les espaces topologiques compacts" [1], P. ALEXANDROFF and P. URYSOHN raised this question for the special case when X is a bicompactum, $X = X_1 \cup X_2$, X_1 closed, X_2 open, $wX_1 = wX_2 = \aleph_0$. Even this special question remained unanswered until 1956, when YU. SMIRNOV [2] proved formula (1) for a local bicompact X and a countable system of arbitrary subspaces X_α, $wW_\alpha = \aleph_0$.

Remark 1. For an arbitrary regular X formula (1) is easily proved in the following two cases:

a) the number of the X_α is finite; they are closed,[1])

b) the number of the X_α is arbitrary; each X_α is dense in X.

Definition. A space X is said to be a *Borel space*, if X is a Borel set (of the classical Hausdorff type $G_{\delta\sigma\delta}\ldots$, of an arbitrary countable ordinal class-number) in some bicompactum $B \supseteq X$.

Theorem 1. *Formula* (1) *holds in full generality* (*that means for an arbitrary number of arbitrary subspaces X_α of X*) *in each of the following two cases*:

(a) *X is a Borel space,*

(b) *X is an arbitrary subset of a perfectly normal bicompactum.*

The theorem of YU. SMIRNOV is obviously contained in our theorem 1.

2. Theorem 2. *Let f be a continuous mapping of a topological space X onto a topological space Y. If the space Y satisfies one of the conditions* (a), (b) *of Theorem* 1, *then*

$$wY \leqq wX .$$

[1]) For a countable system of closed subspaces X_α of a regular X formula (1) does not necessarily hold.

In the special case of bicompact X and Y this theorem has been proved by P. ALEXANDROFF [3].

Corollary. *If under the hypotheses of Theorem 2 the space X is separable metric, then Y is metrizable.*

In this Corollary the assumption that X is separable, is essential. We have, however, the

Theorem 3. *Let Y be metric, while Y satisfies one of the conditions* (a) *or* (b) *of Theorem* 1. *Let $f : X \to Y = fX$ be closed continuous. Then Y is metrizable.*

3. One of the basic tools used for proving the above results, is the notion of a net of a topological space X. I call the system Σ of arbitrary subsets $M \subset X$ a *net* of X if for an arbitrary point $x \in X$ and an arbitrary neighborhood Ox of this point there exists a set $M \in \Sigma$ such that $x \in M \subseteq Ox$.

The following obvious properties of nets are of importance:

1. Let $X = \bigcup_{\alpha} X_{\alpha}$ and Σ_{α} be a net of X_{α}; then $\bigcup_{a} \Sigma_{\alpha}$ is a net of X.

2. Let f be a continuous mapping of X onto Y. This mapping transforms any net of X in a net of Y.

The following Lemma is fundamental for our purposes:

Lemma. *Let $X \subseteq B$, where B is a bicompactum. Let one of the following conditions be satisfied*:

1. *X is a Borel set* (*of any type $G_{\delta\sigma\delta\ldots}$*).
2. *B is perfectly normal.*

Suppose moreover that X contains a net of a cardinality τ. Then there exists an exterior open basis $\mathfrak{B}$ of X with respect to B (*that is a system $\mathfrak{B}$ of open sets $\Gamma \subseteq B$ such that for any $x \in X$ and its neighbourhood Ox in B there exists a $\Gamma \in \mathfrak{B}$ with $x \in \Gamma \subseteq Ox$*) *the cardinality of which does not exceed τ.*

4. The following theorems generalize the known results by P. ALEXANDROFF:

(a) *If the point x of the bicompactum B forms a generalized Borel set $G_{\delta\sigma\delta\ldots}$ of an arbitrary* (*not necessarily countable*) *class-number α, then the character of x in the space B is not greater than the cardinality τ of α.*

(b) *Let X be a Borel space. Then the character and the pseudocharacter of any. point of X are equal.*

The detailed proofs of the results communicated in this report can be found in my Notes [4, 5].

References

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