William Wistar Comfort

Some recent applications of ultrafilters to topology

In: (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part A: Invited papers. Springer, Berlin, 1977. Lecture Notes in Mathematics, 609. pp. [34]--42.

Persistent URL: http://dml.cz/dmlcz/700994

SOME RECENT APPLICATIONS OF ULTRAFILTERS TO TOPOLOGY

W. W. Comfort
Wesleyan University
Middletown, Connecticut, U.S.A.

The three theorems given here hardly exhaust the applications of ultrafilters on ω to topology, of course - indeed, they barely scratch the surface. I have selected them for discussion at this symposium because (in my judgement) they are unusually elegant and pretty results and because they share in common this feature: neither the hypotheses nor the conclusion of these theorems mention ultrafilters, but ultrafilters on ω play a crucial rôle in each of the proofs given here.

These proofs are due to the workers who are identified below. They have circulated informally among aficionadoes, but they have not yet appeared in printed form. I am grateful to these mathematicians for authorizing and encouraging both this brief exposition of their work and the more detailed treatment anticipated in my forthcoming account [5].

§1. *Notation and Terminology.* Throughout these remarks, by a *space* we mean completely regular, Hausdorff space. We denote by ω the least infinite cardinal. The symbol α denotes an (arbitrary) infinite cardinal and the discrete topological space of cardinality α. For X a space we denote by βX the Stone-Čech compactification of X. As is well-known (see for example [12], [6]), $\beta(\alpha)$ may be identified with the space of ultrafilters on α topologized so that $\{\{p \in \beta(\alpha) : A \in p\} : A \subset \alpha\}$ is a base for the closed sets; the inclusion $\alpha \subset \beta(\alpha)$ is effected by identifying the element ξ of α with the principal ultrafilter $\{A \subset \alpha : \xi \in A\}$.

If X and Y are spaces and f is a continuous function from X into Y, we denote by $\overline{f}$ that (unique) continuous function from βX to βY such that $f \subset \overline{f}$. (It may be argued that since $f[X] \subset Y$ for many spaces Y, the function $\overline{f}$ is not well-defined. We hope that in each case our intention concerning Y will be clear; in most cases Y will be taken compact, so that $\overline{f}[X] \subset Y$.)

For $\alpha \geq \omega$ we define

$$U(\alpha) = \{p \in \beta(\alpha) : |A| = \alpha \text{ for all } A \in p\},$$

and we note that $U(\omega) = \beta(\omega) \setminus \omega$.

§2. *A Theorem of Frolík and Kunen.*

A subset A of a space X is said to be *C*-embedded* in X if for every continuous function from A to the space $[0,1]$ (equivalently: to a compact space) there is continuous function g on X such that $f \subset g$.

We state the theorem of Frolík and Kunen.

2.1. *Theorem.* Let X be an infinite compact space in which each infinite, discrete subspace is C^*-embedded. Then X is not homogeneous.

Remarks Concerning the Proof. Suppose we can show that there are p, $q \in U(\omega)$ such that for every $f \in \omega^\omega$ we have $\overline{f}(p) \neq q$ and $\overline{f}(q) \neq p$. Arranging the notation so that $\beta(\omega) \subset X$, we claim that if h is a homeomorphism of X onto X then $h(p) \neq q$. Indeed if $h(p) = q$ then one of these four events occurs: $q \in (h[\omega] \cap \omega)^-$; $q \in (h[\omega] \cap U(\omega))^-$; $p \in (h^{-1}[\omega] \cap \omega)^-$; $p \in (h^{-1}[\omega] \cap U(\omega))^-$. In the first case there is $f \in \omega^\omega$ so that (f agrees with h on $h^{-1}[h[\omega] \cap \omega] \in p$ and) $\overline{f}(p) = q$; in the second case there is a one-to-one function g from ω to $U(\omega)$ such that $g[\omega]$ is discrete and g agrees with h on $h^{-1}[h[\omega] \cap U(\omega)]$, and hence (as is easily shown) there is $f \in \omega^\omega$ such that $\overline{f}(q) = p$; the third case is similar to the first, and the fourth to the second.

To complete the argument, it remains to show that there are p, $q \in U(\omega)$ such that if $f \in \omega^\omega$ then $\overline{f}(p) \neq q$ and $\overline{f}(q) \neq p$. This is a recent result of Kunen [19], recorded also in [6] (Theorem 10.4) and [5], and we shall not repeat it here. Those portions of the argument outlined above had already been supplied by Frolík [8], [9], [10] en route to a number of elegant non-homogeneity results (including the statement that $U(\omega)$ is not homogeneous).

We record some consequences of Theorem 2.1. (A space is said to be *extremally disconnected* if the closure of each of its open subsets is open. A space is an *F-space* if each of its cozero-sets is C^*-embedded.)

2.2. *Corollary.* Let X be an infinite, compact space which satisfies one of the following conditions. Then X is not homogeneous.

(a) X is an F-space;

(b) there is an extremally disconnected space Y such that $X \subset Y$;

(c) there is $\alpha \geq \omega$ such that $X \subset \beta(\alpha)$;

(d) there is a locally compact, σ-compact space Y such that $X = \beta Y \setminus Y$.

Proof. (a) It is easy to show that every countable, discrete subspace of an F-space is C^*-embedded. It is known, more generally, that every countable subspace of an F-space is C*-embedded (see [12] (Problem 14N) or [6] (Lemma 16.15(b))).

(b) βY is extremally disconnected, hence an F-space. Since X is C*-embedded in βY, X is itself an F-space and hence (a) applies.

It is clear that (c) => (b).

Gillman and Henriksen [11] have shown that (d) => (a). An elegant proof, due to Negrepontis, is given in [6] (Lemma 14.16).

The proof of Corollary 2.2 is complete.

We note that Theorem 2.1 was announced (for compact, extremally disconnected spaces) in an editorial footnote appended to [7].

2.3. *A Question.* Let $\alpha \geq \omega$, and let $C(\alpha)$ denote the set of all cardinals γ for which there is $S \subset U(\alpha)$ such that

(1) $|S| = \gamma$, and

(2) if p , $q \in S$ and $p \neq q$, and if $f \in \alpha^{\alpha}$, then $\overline{f}(p) \neq q$ and $\overline{f}(q) \neq p$.

As indicated above, Kunen [19] has shown $2 \in C(\omega)$. In fact, Kunen has shown (without any special assumptions concerning any cardinal numbers) that $2^{\alpha} \in C(\alpha)$. This result suggests the following questions.

Is $(2^{\alpha})^{+} \in C(\alpha)$? Is $\sup C(\alpha) \in C(\alpha)$? Is $2^{2^{\alpha}} = \sup C(\alpha)$? Is $2^{2^{\alpha}} \in C(\alpha)$?

It is shown in [6] and [5] that the answers are affirmative for all α such that $\alpha^{+} = 2^{\alpha}$ or $(2^{\alpha})^{+} < 2^{2^{\alpha}}$.

§ 3. *A Theorem of Ginsburg and Saks.*

We denote by $\{X_i : i \in I\}$ a set of non-empty spaces, for $\emptyset \neq J \subset I$ we write X_J in place of $\prod_{i \in J} X_i$, and we denote by π_J the projection from X_I onto X_J.

We say that a space X is *countably compact* provided that for every $f \in X^{\omega}$ there is $p \in U(\omega)$ such that $\overline{f}(p) \in X$. (In the context of our spaces, this definition agrees with other more usual definitions.)

3.1. *Theorem.* The space X_I is countably compact if and only if X_J is countably compact for all $J \subset I$ such that $0 < |J| \leq 2^{2^{\omega}}$.

Remarks Concerning the Proof. The "only if" implication follows from the fact that the continuous image of a countably compact space is countably compact.

We turn to the "if" implication.

Following Bernstein [2], we say (for $p \in U(\omega)$) that a space X is *p-compact* if $\overline{f}(p) \in X$ for every $f \in X^{\omega}$. It is not difficult to show (cf. [2] or [13]) that the product of p-compact spaces is p-compact. Indeed we have the following statement:

(*) If $f \in (X_I)^{\omega}$ and if there is $x = \langle x_i : i \in I \rangle \in X_I$ such that $(\pi_i \circ f)^{-}(p) = x_i$ for all $i \in I$, then $\overline{f}(p) = x$.

It follows that if the desired conclusion fails then there are $f \in (X_I)^{\omega}$, and (for every $p \in U(\omega)$) an element $i(p)$ of I, such that $(\pi_{i(p)} \circ f)^{-}(p) \notin X_{i(p)}$. Then with $J = \{i(p) : p \in U(\omega)\}$ we have $\pi_J \circ f \in (X_J)^{\omega}$ and $(\pi_J \circ f)^{-}[U(\omega)] \cap X_J = \emptyset$, so that X_J is not countably compact. Since $|J| \leq |U(\omega)| = 2^{2^{\omega}}$, the proof is complete.

We remark that (*) may be established directly, as in [5], or by appeal to the work of Glicksberg [15]. It is noted in [15] that a product space X_I is pseudocompact if and only if X_J is pseudocompact for all $J \subset I$ such that $0 < |J| \leq \omega$; thus our space X_I is pseudocompact. It then follows from Theorem 1 of [15] that $\beta(X_I) = \prod_{i \in I} \beta X_i$; statement (*) is then obvious.

The technique used in the proof just given has been used by Ginsburg and Saks [13] and Saks [20] in connection with product-space theorems concerning a multitude of topological properties.

Theorem 3.1 has been proved in [4] and [20], and in [13] in the case that the spaces X_i are pairwise homeomorphic.

3.2. *A Question.* The following question, raised in [4], apparently remains unsolved. We emphasize that, as with Question 2.3 above, a solution is desired in ZFC (without the assumption of special set-theoretic axioms).

Is the cardinal number $2^{2^{\omega}}$ optimal in Theorem 3.1? Is there a family $\{X_i : i \in I\}$ of spaces, with $|I| = 2^{2^{\omega}}$, such that X_I is not countably compact but X_J is countably compact for all $J \subset I$ such that $\emptyset \neq J \neq I$? Is there a space X such that $X^{2^{2^{\omega}}}$ is not countably compact but X^{α} is countably compact for all cardinals $\alpha < 2^{2^{\omega}}$?

Reference is made in [5] to published and forthcoming works of van Douwen, of Juhász, Nagy and Weiss, of Kunen, of Rajagopalan, of Rajagopalan and Woods, and of Vaughan, which answer portions of these questions under a variety of set-theoretic assumptions known to be consistent with ZFC.

§4. *On Glazer's Proof of Hindman's Theorem.*

If S is a set we write

$$[S]^{\omega} = \{A \subset S : |A| = \omega\} \quad \text{and}$$

$$[S]^{<\omega} = \{A \subset S : |A| < \omega\}.$$

We denote the set of positive integers by $\mathbb{N}$, and for $F = \{k_n : n < m\} \in [\mathbb{N}]^{<\omega}$ we set $\Sigma F = \Sigma\{k_n : n < m\}$.

It was conjectured by Graham and Rothschild [16] that if $\mathbb{N} = A_0 \cup A_1$ then there are $k \in \{0,1\}$ and $B \in [A_k]^{\omega}$ such that $\Sigma F \in A_k$ for all $F \in [B]^{<\omega}$. The following statement, due to Hindman [17], establishes (a statement formally stronger than) the Graham-Rothschild conjecture. Hindman's proof [17] makes no use of ultrafilters, and will not concern us here. The proof we shall discuss is due to Glazer.

4.1. *Theorem.* If $n < \omega$ and $\mathbb{N} = \bigcup_{k<n} A_k$, then there are $k < n$ and $B \in [A_k]^{\omega}$ such that $\Sigma F \in A_k$ for all $F \in [B]^{<\omega}$.

Remarks Concerning the Proof. Define

$$\mathcal{A} = \{A \subset \mathbb{N} : \text{there is } B \in [A]^{\omega} \text{ such that } \Sigma F \in A \text{ for all } F \in [B]^{<\omega}\}.$$

It is enough to show that there is $p \in \beta(\mathbb{N})$ such that $p \subset \mathcal{A}$. To this end, for $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ define

$$A - n = \{k \in \mathbb{N} : k + n \in A\}$$

and define an operation $\dot{+}$ on $\beta(\mathbb{N})\times\beta(\mathbb{N})$ by

$$p \dot{+} q = \{A \subset \mathbb{N} : \{n \in \mathbb{N} : A - n \in p\} \in q\}.$$

It is easy to show that $\dot{+}$ is an associative function into $\beta(\mathbb{N})$ and that the function $q \to p \dot{+} q$ is, for each $p \in \beta(\mathbb{N})$, continuous as a function of q. It follows from Zorn's Lemma that there is $p \in \beta(\mathbb{N})$ such that $p \dot{+} p = p$; since $\dot{+}$ extends the usual addition function of $\mathbb{N}$, and since there is no $n \in \mathbb{N}$ such that $n + n = n$, we have $p \in \beta(\mathbb{N})\setminus\mathbb{N}$.

We outline the proof that $p \subset A$. For $A \in p$ define

$$A^* = \{k \in \mathbb{N} : A - k \in p\},$$

and now fix $A = A_0 \in p$. Choose $k_0 \in A_0^* \cap A_0$ and recursively define $A_{n+1} = (A_n - k_n) \cap A_n$ and choose $k_{n+1} \in A_{n+1}^* \cap A_{n+1}$ so that $k_{n+1} > k_n$. Finally, define $B = \{k_n : n < \omega\}$. It is easily shown that $\Sigma F \in A_0 = A$ for all $F \in [B]^{<\omega}$.

Additional details of this proof are available in [14] and [5].

It is appropriate to note that it was F. Galvin who first raised the question whether there is $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$ such that $\{n \in \mathbb{N} : A - n \in p\} \in p$ whenever $A \in p$, and who pointed out that an affirmative response would serve to establish the Graham-Rothschild conjecture; it was Glazer who defined the function $\dot{+}$ and showed the existence of $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$ such that $p = p \dot{+} p$.

It has been pointed out to me by I. Prodanov and others that a proof which has been available for some years of Ramsey's theorem $\omega \to (\omega)_n^2$ is reminiscent of the proof just given. Indeed let $p \in U(\omega)$ and suppose that $[\omega]^2 = \bigcup_{k<n} A_k$. For $i < \omega$ define $A_k(i) = \{m < \omega : \{m,i\} \in A_k\}$ and set $B_k = \{i : A_k(i) \in p\}$. Since $\bigcup_{k<n} B_k = \omega$ there is $B_{\bar{k}} \in p$. Choose $k_0 \in B_{\bar{k}}$ and recursively choose $k_{m+1} \in B_{\bar{k}} \cap \bigcap_{j\leq m} A_{\bar{k}}(k_j)$ so that $k_{m+1} > k_m$. Then $B = \{k_m : m < \omega\}$ satisfies $B \in [\omega]^\omega$ and $[B]^2 \subset A_{\bar{k}}$. This is essentially the proof of the relation $\omega \to (\omega)_n^2$ given, for example, by Chang and Keisler [3] (Theorem 3.3.7) and by Jech [18] (Problem 7.5.1).

4.2. *Hindman's Theorem in* ZF. A mathematician reading Glazer's proof of Hindman's theorem may be dissatisfied because the proof appeals to the Axiom of Choice while the result itself "looks as if" it should be provable without appeal to that axiom. The same objection may be lodged against the original proof of Hindman [17] and the proof given subsequently by Baumgartner [1].

Responding to an inquiry whether Hindman's Theorem is a theorem in ZF, Professor Baumgartner has communicated the following information (letter of September, 1976); this is recorded here with his kind permission.

Hindman's Theorem is a Π_2^1 assertion, *i.e.*, it can be put in the form $(\forall X \in \mathcal{P}(\omega))(\exists Y \in \mathcal{P}(\omega))\phi(X,Y)$, where ϕ contains only first-order quantifiers ranging over the natural numbers. The following theorem, due to Shoenfield, asserts that Π_2^1 sentences are "absolute".

Theorem. Let M and N be transitive models of ZF (N may be the universe of set theory) such that $M \subset N$ and every countable ordinal of N lies in M. Then if ϕ is any Π^1_2-sentence, ϕ is true in M if and only if ϕ is true in N.

The standard reference is [21] (pp. 132-139), though the result given there is weaker than the theorem above.

To verify Hindman's Theorem it is enough to show that if ϕ is Π^1_2 and ZFC $\vdash \phi$, then ZF $\vdash \phi$. Suppose that ϕ is $(\forall X)(\exists Y)\psi(X,Y)$. Fix X and consider L[X], the class of all sets constructible from X. It is well-known that $L[X] \models \phi$. Hence $\exists Y \in L[X]$ such that $L[X] \models \phi(X,Y)$. But since $\psi(X,Y)$ involves only first-order quantification over the integers, and since the integers are the same in L[X] as in the "real world", it follows that $\psi(X,Y)$ is really true. Thus we have shown that $(\forall X)(\exists Y)\psi(X,Y)$, *i.e.*, that ϕ is true. Since this argument will work in any model of ZF, it follows that ZF $\vdash \phi$.

LIST OF REFERENCES

[1] James Baumgartner, *A short proof of Hindman's theorem*, J. Combinatorial Theory A 17 (1974), 384-386.

[2] Allen R. Bernstein, *A new kind of compactness for topological spaces*, Fundamenta Math. 66 (1970), 185-193.

[3] C. C. Chang and H. J. Keisler, Studies in Model Theory. Studies in Logic and the Foundations of Mathematics Vol. 73. North-Holland Publ. Co., Amsterdam-London, 1973.

[4] W. W. Comfort, Review of [13], Mathematical Reviews 52 #1633 (1976), 227-228.

[5] W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. (1977), (to appear).

[6] W. W. Comfort and S. Negrepontis, The Theory of Ultrafilters, Grundlehren der math. Wissenschaften vol. 211. Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[7] B. A. Efimov, *On the embedding of extremally disconnected spaces into bicompacta*, In: General Topology and Its Relations to Modern Analysis and Algebra III. Proc. Third Prague Topological Symposium, Prague, 1971, edited by J. Novák, pp. 103-107. Academia Prague, Prague, 1972.

[8] Zdeněk Frolík, *Types of ultrafilters on countable sets*, In: General Topology and its Relations to Modern Analysis and Algebra II, Proc. Second Prague Topological Symposium, Prague, 1966, edited by J. Novák, pp. 142-143. Academia Prague, Prague, 1967.

[9] Zdeněk Frolík, *Homogeneity problems for extremally disconnected spaces*, Commentationes Math. Univ. Carolinae 8 (1967), 757-763.

[10] Z. Frolík, *Maps of extremally disconnected spaces, theory of types, and applications*, In: General Topology and its Relations to Modern Analysis and Algebra, Proc. (1968) Kanpur Topological Conference, pp. 131-142. Academia Prague, Prague, 1971.

[11] L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. 82 (1956), 366-391.

[12] Leonard Gillman and Meyer Jerison, Rings of Continuous Functions, D. Van Nostrand Co., Inc., Princeton, 1960.

[13] John Ginsburg and Victor Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. 57 (1975), 403-418.

[14] Steven Glazer, Manuscript, (to appear).

[15] Irving Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. 90 (1959), 369-382.

[16] R. L. Graham and B. L. Rothschild, *Ramsey's theorem for n-parameter sets*, Trans. Amer. Math. Soc. 159 (1971), 257-292.

[17] Neil Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Combinatorial Theory (A) 17 (1974), 1-11.

[18] Thomas J. Jech, The Axiom of Choice. Studies in Logic and the Foundations of Mathematics vol. 75. North-Holland Publ. Co., Amsterdam-London, 1973.

[19] Kenneth Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. 172 (1972), 299-306.

[20] Victor Saks, *Ultrafilter invariants in topological spaces*, Manuscript, (to appear).

[21] J. R. Shoenfield, *The problem of predicativity*, In: Essays on the Foundations of Mathematics, pp. 132-139. Magnes Press, Hebrew University, Jerusalem, 1966.