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STABILITY OF BANACH ALGEBRAS

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Given a mathematical structure, it is of interest mathematically to know whether other structures 'near' to it in some sense share various properties of the original structure. If the structure is used as part of a mathematical model of some physical situation then this question is particularly important as the parameters determining the system cannot be evaluated with complete accuracy. Thus it will not be known exactly which of a number of systems is involved and so properties which are common to all these systems have a special significance.

We shall consider a Banach algebra $\mathfrak{A}$ with multiplication π. Juxtaposition of elements of $\mathfrak{A}$ will denote their π product. We shall also consider other multiplications, that is continuous associative bilinear maps ρ; $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. We assume $\|ab\| \leq \|a\|\|b\|$ but shall not assume $\|\rho(a,b)\| \leq \|a\|\|b\|$ $(a,b \in \mathfrak{A})$. The strongest possible result is that if ρ is sufficiently near π then $(\mathfrak{A},\pi)$ and $(\mathfrak{A},\rho)$ are isomorphic, that is there is a continuous linear bijection T; $\mathfrak{A} \to \mathfrak{A}$ with $T\rho(a,b) = (Ta)(Tb)$ $(a,b \in \mathfrak{A})$. This is equivalent to saying that ρ has the form

$$\rho(a,b) = T^{-1}((Ta)(Tb)) \qquad \text{(i)}.$$

It is easy to see that defining ρ by formula (i) gives a multiplication on $\mathfrak{A}$ so our question is whether (i) is a necessary as well as a sufficient condition for ρ to be a multiplication. This question was first raised by J.L. Taylor and the result we give below was also obtained by Raeburn and Taylor [10]. As in [3] we write $\mathfrak{L}^n(\mathfrak{A})$ for the Banach space of continuous n linear functions with variables and values in $\mathfrak{A}$.

DEFINITION 1. A Banach algebra $\mathfrak{A}$ is <u>stable</u> if there exists $\varepsilon > 0$ such that for each multiplication ρ on $\mathfrak{A}$ with $\|\rho-\pi\| < \varepsilon$ there is a non-singular element T of $\mathfrak{L}^1(\mathfrak{A})$ with $\rho(a,b) = T^{-1}(T(a)T(b))$ $(a,b \in \mathfrak{A})$. ($\|\rho-\pi\|$ denotes the $\mathfrak{L}^2(\mathfrak{A})$ norm.)

To motivate our result consider the related problem of representing the multiplications in a one parameter family $\{\rho_t\}$, where $\rho_0 = \pi$, in the form $\rho_t(a,b) = T_t^{-1}(T_t a)(T_t b)$ for a suitably chosen one parameter family T_t from $\mathfrak{L}^1(\mathfrak{A})$. If

$$\left(\frac{d\rho_t}{dt}\right)_{t=0} = S$$

then differentiating the associative law

$$\rho_t(a,\rho_t(b,c)) = \rho_t(\rho_t(a,b),c)$$

and putting t = o gives

$$S(a,bc) + a\,S(b,c) = S(ab,c) + S(a,b)c$$

that is

$$a\,S(b,c) - S(ab,c) + S(a,bc) - S(a,b)c = 0 \qquad \text{(ii)}.$$

Putting $R = (\frac{dT_t}{dt})_{t=0}$, differentiating the relation $T_t\rho_t(a,b) = T_taT_tb$ and putting $t = 0$ gives

$$a\,R(b) - R(ab) + R(a)b = S(a,b) \qquad \text{(iii)}.$$

Thus under suitable conditions of differentiability we must solve (iii) for R given that S satisfies (ii). For any $R \in \mathfrak{L}^1(\mathfrak{A})$ (resp. $S \in \mathfrak{L}^2(\mathfrak{A})$) the left hand side of (iii) (resp. (ii)) defines an element δ^2R of $\mathfrak{L}^2(\mathfrak{A})$ (resp. δ^3S of $\mathfrak{L}^3(\mathfrak{A})$). It is easy to see, by direct substitution, that $\delta^3\delta^2R = 0$ so the question of solving (iii) given (ii) is just the question of whether $\ker\delta^3 = \operatorname{im}\delta^2$. This has been considered extensively (see [3] and elsewhere).

THEOREM 2. Let $\mathfrak{A}$ be a Banach algebra with $\ker\delta^3 = \operatorname{im}\delta^2$ and $\operatorname{im}\delta^3$ closed in $\mathfrak{L}^3(\mathfrak{A})$. Then $\mathfrak{A}$ is stable.

Proof. Let $\epsilon > 0$ and let ρ be a multiplication on $\mathfrak{A}$ with $\|\pi-\rho\| < \epsilon$. By the open mapping theorem and the hypotheses that $\operatorname{im}\delta^2 = \ker\delta^3$, a closed subspace of $\mathfrak{L}^2(\mathfrak{A})$, and $\operatorname{im}\delta^3$ is closed, there exist $K,L > 0$ such that if $S \in \ker\delta^3$ and $T \in \mathfrak{L}^2(\mathfrak{A})$ there exist $R \in \mathfrak{L}^1(\mathfrak{A})$ and $S' \in \mathfrak{L}^2(\mathfrak{A})$ with $\delta^2R = S$, $\|R\| \le K\|S\|$, $\delta^3S' = \delta^3T$ and $\|S'\| \le L\|\delta^3T\|$.

We have

$$\begin{aligned}(\pi-\rho)((\pi-\rho)(a,b),c) - (\pi-\rho)(a,(\pi-\rho)(b,c)) &= \\ (ab)c - a(bc) + \rho(\rho(a,b),c) - \rho(a,\rho(b,c)) & \\ - \rho(ab,c) - \rho(a,b)c + \rho(a,bc) + a\rho(b,c) & \\ = (\delta^3\rho)(a,b,c) = \delta^3(\rho-\pi)(a,b,c) &\end{aligned}$$

so that $\|\delta^3(\rho-\pi)\| \le 2\epsilon^2$. Thus there is $S \in \mathfrak{L}^2(\mathfrak{A})$ with $\|S\| \le 2L\epsilon^2$ and $\delta^3S = \delta^3(\rho-\pi)$. As $\rho-\pi-S \in \ker\delta^3$ there is $R \in \mathfrak{L}^1(\mathfrak{A})$ with $\delta^2R = \rho-\pi-S$ and $\|R\| \le K\|\rho-\pi-S\| \le K(\epsilon+2L\epsilon^2) = K\epsilon(1+2L\epsilon) < 2K\epsilon < \frac{1}{2}$ provided $\epsilon < \text{Max}\ ((2L)^{-1},(4K)^{-1})$. Thus I+R is regular in $\mathfrak{L}^1(\mathfrak{A})$. Putting $\rho'(a,b) = (I+R)^{-1}[(I+R)a(I+R)b]$ and expanding $\rho(a,b) - \rho'(a,b)$ in powers of R, the constant and first degree terms are

$$\begin{aligned}\|\rho(a,b) - ab + R(ab) - aR(b) - R(a)b\| &= \|(\rho-\pi-\delta^2R)(a,b)\| \\ &= \|S(a,b)\| \\ &\le 2L\epsilon^2\|a\|\|b\|\end{aligned}$$

whereas the other terms give

$$\|\sum_{n=0}^{\infty}(-1)^n\,R^{n+2}(ab) - R^{n+1}(aRb) - R^{n+1}((Ra)b) + R^n(RaRb)\|$$

$$\le \sum_{n=0}^{\infty} 4\|R\|^{n+2}\|a\|\|b\|$$

$$\le 4\|R\|^2(1-\|R\|)^{-1}\|a\|\|b\|$$

$$\leq 32K^2\epsilon^2\|a\|\|b\| \ .$$

Thus $\|\rho-\rho'\| \leq (32K^2+2L)\epsilon^2$. Now put $\rho_1(a,b) = (I+R)\rho((I+R)^{-1}a, (I+R)^{-1}b)$. We have

$$\|(\rho_1-\pi)(a,b)\| = \|(I+R)(\rho-\rho')[(I+R)^{-1}a, (I+R)^{-1}b]\|$$

so that

$$\begin{aligned}\|\rho_1-\pi\| &\leq \|I+R\|\|\rho-\rho'\|\|(I+R)^{-1}\|^2 \\ &\leq 8(32K^2+2L)\epsilon^2 \\ &= M\epsilon^2\end{aligned}$$

say. As ρ_1 is a multiplication we can apply the above argument with ρ replaced by ρ_1 obtaining R_1, S_1, ρ_2 and so on inductively. At the nth stage we have $\|\rho_n-\pi\| \leq M^{2^n-1}\epsilon^{2^n}$. Thus if $\epsilon < M^{-1}$ we have $\rho_n \to \pi$ as $n \to \infty$. Putting $W_n = (I+R_n)\ldots(I+R_1)(I+R)$ we have, since $\|R_n\| < 2KM^{2^n-1}\epsilon^{2^n}$, $W = \lim W_n$ exists and is regular because $\|W_n^{-1}\| \leq [(1-\|R\|)(1-\|R_1\|)\ldots(1-\|R_n\|)]^{-1}$ where the infinite product $\Pi(1-\|R_i\|)$ is convergent to a non-zero limit. Thus

$$ab = \lim_n \rho_n(a,b) = \lim_n W_n\rho(W_n^{-1}a, W_n^{-1}b) = W\rho(W^{-1}a, W^{-1}b) \ .$$

Replacing a, b by Wa, Wb we get $\rho(a,b) = W^{-1}(Wa)(Wb)$.

The following Banach algebras are stable

(i) The algebra C(X) of all bounded continuous complex valued functions on a topological space X.

(ii) The algebra $\mathcal{L}(\mathfrak{X})$ of all bounded linear operators on a Banach space $\mathfrak{X}$.

(iii) The algebra $\mathcal{LC}(\mathfrak{X})$ of all compact operators on a Banach space $\mathfrak{X}$ with an unconditional basis.

(iv) The algebra C_1 of trace class operators on a separable Hilbert space.

(v) The sequence algebras ℓ^1 and ℓ^∞ with pointwise multiplication.

(vi) Any type I or hyperfinite von Neumann algebra.

(vii) The group algebras $L^1(G)$, $M(G)$ of an amenable locally compact group G.

By contrast the following Banach algebras are not stable

(i)′ The algebra C_2 of Hilbert-Schmidt operators and, more generally, the von Neumann-Schatten classes C_p [1; p.1089] for $1 < p < \infty$.

(ii)′ The sequence algebra ℓ^p with pointwise multiplication for $1 < p < \infty$.

(iii)′ The group algebra $\ell^1(F_2)$ of the free group on 2 generators.

The positive results, with the exception of (i) (see Theorem 3 below) are all proved by showing that the hypotheses of Theorem 2 are satisfied. For (ii) this is a result of Kaliman and Selivanov [8], for (iii) the proof is a minor adaptation of the Hilbert space case [4; p.697], (vi) depends on results of Kadison and Ringrose ([6; Theorem 4.4] and [7; Theorem 3.1]) and (vii) follows from [3; Theorem 2.5 and Proposition 1.9] and [4; Theorem 4.4 and Example 4.2]; (iv) and (v) appear in [5;

§3B and 3C]. The counterexamples also appear in [5].

THEOREM 3. Let X be a topological space. Then C(X) is stable.

Proof. Since C(X) is a commutative C*-algebra it is (isometrically isomorphic with) $C(\Omega)$ for some compact Hansdorff space Ω. Thus the fact that $\ker \delta^3 = \operatorname{im} \delta^2$ is a result of Kamowitz [9; Theorem 4.7]. To show that the other hypothesis is satisfied we consider first the case in which X is a metric space with metric d. We shall show $\operatorname{im} \delta^3$ is closed by showing $\operatorname{im} \delta^3 = \ker \delta^4$ where δ^4; $\mathcal{L}^3(\mathfrak{A}) \to \mathcal{L}^4(\mathfrak{A})$ is defined by

$$(\delta^4 T)(a,b,c,d) = aT(b,c,d) - T(ab,c,d) + T(a,bc,d) - T(a,b,cd) + T(a,b,c)d .$$

For each positive integer k define f_k; $[0,\infty) \to [0,1]$ by

$$f_k(t) = 0 \qquad 0 \le t \le 2^{-k-1} \qquad \text{or } 2^{-k+1} \le t$$

$$f_k(2^{-k}) = 1$$

and f_k is linear on $[2^{-k-1}, 2^{-k}]$ and $[2^{-k}, 2^{-k+1}]$. We define f_0 in the same way on $[0,1]$ and put $f_0(t) = 1$ for $t \ge 1$. For $x \in X$ let $g_{x,k}(y) = (f_k \circ d(x,y))^{\frac{1}{2}}$. As in Helemskii [2; §4] if we put $J_x(a) = \sum_k g_{x,k} \otimes g_{x,k}\, a$ then J_x is a map $M_x (= \{a; a \in C(X), a(x) = 0\}) \to M_x \hat{\otimes} M_x$ of norm at most 2. If $T \in \ker \delta^4$ put

$$S(a,b)(x) = -T(J_x(a-a(x)1), b - b(x)1)(x)$$
$$+ a(x)T(1,1,b)(x) - b(x)T(a-a(x)1,1,1)(x)$$

where we have also used T to denote the map $M_x \hat{\otimes} M_x \times C(X)$ derived from T. To see that for each a,b, $S(a,b) \in C(X)$ it is enough to show that $x \to J_x(a-a(x)1)$ is continuous $X \to C(X) \hat{\otimes} C(X)$ for each $a \in C(X)$. Let $y \in X$. Since $J_x(a-a(x)1)$ is linear in a and $\|J_x(a-a(x)1)\| \le 4\|a\|$ it is enough to prove continuity at y for a in a dense subspace of C(X). Accordingly we suppose that a is constant in a neighbourhood $\{z; d(z,y) \le 2^{-n}\}$ of y. If $d(y,z) < 2^{-n-1}$ then $(a - a(z)1)(w) = 0$ for $d(z,w) < 2^{-n-1}$ so for such z

$$J_z(a-a(z)1) = \sum_{k=0}^{n+2} g_{z,k} \otimes g_{z,k}(a-a(z)1)$$

and the map $x \to g_{x,k}$ is continuous $X \to C(X)$ for each k.

Finally we show $\delta^3 S(a,b,c)(x) = T(a,b,c)(x)$. First consider the case $a,c \in M_x$. Then

$$\begin{aligned}\delta^3 S(a,b,c)(x) &= -S(ab,c)(x) + S(a,bc)(x)\\ &= T(J_x(ab),c)(x) - T(J_x(a),bc)(x)\\ &= T(a,b,c)(x)\end{aligned}$$

because

$$\begin{aligned}0 &= \delta^4 T(g_{x,k}, g_{x,k} a, b, c)(x)\\ &= -T(g_{x,k}{}^2 a, b, c)(x) + T(g_{x,k}, g_{x,k} ab, c)(x)\end{aligned}$$

$$-T(g_{x,k}, g_{x,k}a, bc)(x)$$

so that summing over k and using $\sum_k g_{x,k}(y)^2 = 1$ $(y \neq x)$ we obtain the required relationship.

Now suppose $a = 1$. Then

$$\begin{aligned}\delta^3 S(1,b,c)(x) &= S(1,bc)(x) - S(1,b)(x)\ c(x) \\ &= T(1,1,bc)(x) - T(1,1,b)(x)\ c(x) \\ &= T(1,b,c)(x)\end{aligned}$$

by considering the identity $(\delta^4 T)(1,1,b,c) = 0$.

Last consider the case $a \in M_x$, $c = 1$.

$$\begin{aligned}\delta^3 S(a,b,c)(x) &= -S(ab,1)(x) \\ &= T(ab,1,1)(x) \\ &= T(a,b,1)(x)\end{aligned}$$

by considering the equation $\delta^4 T\ (a,b,1,1)(x) = 0$.

To treat the case of general X rather more attention to the constants K, L, M of Theorem 2 is necessary. By the proof of Kamovitz' theorem that $\ker \delta^3 = \operatorname{im} \delta^2$ given in [3; Proposition 8.2] we see $K = 1$. From the definition of S in terms of T above we see $\|S\| \le 11\|T\|$ so for metric X we can take $L = 11$ giving $M = 432$. Thus, using the notation of Theorem 2 applied to this case

$$\begin{aligned}\|W_n - I\| &\le (1+\|R\|)(1+\|R_1\|)\ \dots\ (1+\|R_n\|) - 1 \\ &\le (1+2\varepsilon)(1+2M\varepsilon^2)\ \dots\ (1+2M^{2^{n-1}}\varepsilon^{2^n}) - 1 \\ &\le \frac{M\varepsilon}{1-M\varepsilon}\end{aligned}$$

so $\|W-I\| < M\varepsilon(1-M\varepsilon)^{-1}$. From this we see $\|W^{-1}-I\| \le 2M\varepsilon(1-M\varepsilon)^{-1}$. If $\varepsilon < (3M)^{-1}$ we get $\|W-I\| < \frac{1}{2}$ and $\|W^{-1}-I\| < 1$.

For each finite subset F of C(X) let $\mathfrak{A}_F$ be the smallest closed unital * subalgebra of C(X) containing F which is ρ closed. $\mathfrak{A}_F$ is separable as it is the closed linear span of elements which can be expressed by a finite number of applications of $\pi, \rho, *$ to elements of F and these form a denumerable set. Hence $\mathfrak{A}_F$ is isomorphic with $C(\Omega)$ for a compact metric space Ω. Thus by the theorem for metric X, if ρ is a multiplication on C(X) with $\|\rho-\pi\| < 1/1296$ then for each finite $F \subseteq C(X)$ there is an invertable element W_F of $\mathfrak{L}^1(\mathfrak{A}_F)$ with $\rho(a,b) = W_F^{-1}(W_F(a)W_F(b))$ $(a,b \in \mathfrak{A}_F)$ and $\|W_F - I_{\mathfrak{A}_F}\| < \frac{1}{2}$, $\|W_F^{-1} - I_{\mathfrak{A}_F}\| < 1$. Suppose F,G are two finite subsets of C(X) with $\mathfrak{A}_F \subseteq \mathfrak{A}_G$ so that for $a,b \in \mathfrak{A}_F$, $W_F^{-1}(W_F a W_F b) = W_G^{-1}(W_G a W_G b)$, that is $\alpha = W_G W_F^{-1}$ is an isomorphism of $\mathfrak{A}_F$ into $\mathfrak{A}_G$ with $\|\alpha - \iota\| < \frac{1}{2}$ where ι is the injection $\mathfrak{A}_F \subseteq \mathfrak{A}_G$. If φ is a multiplicative linear functional on $\mathfrak{A}_G$ then $\alpha^*\varphi$ and $\iota^*\varphi$ are multiplicative linear functionals on $\mathfrak{A}_F$ with $\|\alpha^*\varphi - \iota^*\varphi\| < 2$ so $\alpha^*\varphi = \iota^*\varphi$. Hence $\varphi(\alpha a) = \varphi(a)$ for all $a \in \mathfrak{A}_F$ and all φ. This implies that $a = \alpha a$ and hence $W_G a = W_F a$ for all a in $\mathfrak{A}_F$. The map W on C(X) defined by

$$Wa = W_F a \qquad \text{if } a \in \mathfrak{A}_F$$

is thus well defined, bounded, linear and has $\|W-I_{C(X)}\| \leq \frac{1}{2}$ so that it is invertible. On each $\mathfrak{A}_F$, $W^{-1}a = W_F^{-1}a$ so that $\rho(a,b) = W^{-1}(WaWb)$ $a,b \in C(X)$.

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