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SOME PROBLEMS CONCERNING C(X)

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0. In this paper, some existence problems are investigated concerning function classes having the form $C(X)$, $C(Y)|X = \{f|X: f \in C(Y)\}$ or $C(Y) \circ p = \{f \circ p: f \in C(Y)\}$ where $p: X \to Y$ is a given map. The following questions are treated. Given a set X and a class Φ of real-valued functions defined on X, is there a topology on X such that $\Phi = C(X)$ (or on a set Y such that $\Phi = C(Y) \circ p$ for some surjective map $p: X \to Y$, or on a set $Y \supset X$ such that $\Phi = C(Y)|X$)? Given a topological space X and a Φ as above, is there a space $Y \supset X$ containing X as a subspace such that $\Phi = C(Y)|X$? Given a ring A, is there a topological space X (or a topological space Y and a subspace $X \subset Y$) such that A be isomorphic with $C(X)$ (or $C(Y)|X$)?

In answering (or partially answering) questions of this kind, we will use some types of function classes defined in [5] and [14].

1. Let X be an arbitrary non-empty set. Φ is said to be a function class on X if each $f \in \Phi$ is a real-valued function $f: X \to R$ defined on X. If $S \neq \emptyset$ and $p: S \to X$, we denote $\Phi \circ p = \{f \circ p: f \in \Phi\}$; for $\emptyset \neq T \subset X$, define $\Phi | T = \{f|T: f \in \Phi\}$. A function ring or a function lattice on X is a function class on X that is a ring or a lattice, respectively, under pointwise defined operations $+$, $\cdot$, $\vee$, $\wedge$, and that contains all constant functions. An affine lattice on X is a function lattice Φ on X such that $f \in \Phi$, $c \in R$ implies $f + c \in \Phi$, $cf \in \Phi$.

If X is a topological space with topology τ, $C(X)$ or $C(\tau)$ denotes the class of all continuous real-valued functions on X. Clearly $C(X)$ is a function ring and an affine lattice on X, and the same holds for $C(Y)|X$ or $C(Y) \circ p$ with $p: X \to Y$.

A function class Φ on X is said to be strongly composition-closed (scc), or composition-closed (cc) ([5], pp. 143 and 146), or weakly composition-closed (wcc) ([14], p. 114) if the following is true: given a family $\{f_i: i \in I\} \subset \Phi$, consider the map

(1.1) $\quad h: X \to R^I, \quad h(x) = (f_i(x)) \quad \text{for} \quad x \in X,$

and a function $k \in C(h(X))$, or $k \in C(\overline{h(X)})$, or $k \in C(R^I)$ respectively; then $k \circ h \in \Phi$. Here R^I is equipped with the product topology, $\bar{Z}$ denotes the closure with respect to this topology, $h(X)$ and $\overline{h(X)}$ are considered as subspaces of R^I.

If the above conditions are restricted by the assumption that the index set I is countable, then Φ is said to be <u>countably strongly composition-closed</u> (cscc), <u>countably composition-closed</u> (ccc), <u>countably weakly composition-closed</u> (cwcc) respectively. Similarly, if I is supposed to be finite, Φ is said to be <u>finitely strongly composition-closed</u> (fscc), <u>finitely composition-closed</u> (fcc), or <u>finitely weakly composition-closed</u> (fwcc). For these classes, the following implications hold ([5], p. 147):

$$\begin{array}{ccccc} \text{scc} & \Rightarrow & \text{cc} & \Rightarrow & \text{wcc} \\ \Downarrow & & \Downarrow & & \Updownarrow \\ \text{cscc} & \Rightarrow & \text{ccc} & \Leftrightarrow & \text{cwcc} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{fscc} & \Rightarrow & \text{fcc} & \Leftrightarrow & \text{fwcc} \end{array}$$

Φ is said to be <u>inversion-closed</u> if $f \in \Phi$, $f(x) \neq 0$ for $x \in X$ implies $1/f \in \Phi$.

If $f: X \to R$, let us denote $Z(f) = \{x \in X: f(x) = 0\}$; $Z(f)$ is the <u>zero-set of the function</u> f. If Φ is a function class on X, denote $Z(\Phi) = \{Z(f): f \in \Phi\}$. If X is a topological space with topology τ, we write $Z(X) = Z(C(X))$ or $Z(\tau) = Z(C(\tau))$; the elements of $Z(X)$ are the <u>zero-sets of the space</u> X. For $f: X \to R$, $c \in R$, put $X(f \geq c) = \{x \in X: f(x) \geq c\}$, $X(f \leq c) = \{x \in X: f(x) \leq c\}$. The sets $X(f \geq c)$ and $X(f \leq c)$ are called <u>level sets</u> (or <u>Lebesgue sets</u>) of f.

A function class Φ on X is said to be <u>complete</u> (or <u>uniformly closed</u>) if $f_n \in \Phi$ implies $f \in \Phi$ whenever $f_n \to f$ uniformly on X.

<u>2</u>. We start by examining the first type of questions formulated in the introduction. Let Φ be a function class on a set X. Under what conditions does a topology exist on the set X such that $\Phi = C(X)$? This question is answered by the following

<u>Theorem 1</u>. If X is a topological space and $\Phi = C(X)$, then

(a) Φ is a function ring,

(a') Φ is an affine lattice,

(b) $f = \{\sup g_i: i \in I\} = \inf\{h_j: j \in J\}$, $g_i, h_j \in \Phi$, $I \neq \emptyset$, $J \neq \emptyset$ implies $f \in \Phi$,

(c) $(f \vee (-c)) \wedge c \in \Phi$ for $c \in R$ implies $f \in \Phi$, (c') Φ is inversion-closed.

Conversely if Φ is a function class on X satisfying either (a), (b), (c), or (a), (b), (c'), or (a'), (b), (c), then there is a topology on X such that $\Phi = C(X)$.

Proof. The necessity of (a) to (c') can be easily checked (see [4], pp. 184-185). If Φ satisfies (b) and either (a) or (a'), then the same holds for the class Φ^* composed of all bounded functions in Φ; hence, by [4], Theorem 8, $\Phi^* = C^*(\tau)$ for some topology τ on X. In the case of (a') we easily get that $f \in \Phi$ implies $f_c = (f \vee (-c)) \wedge c \in \Phi^* = C^*(\tau) \subset C(\tau)$, hence $f \in C(\tau)$; conversely $f \in C(\tau)$ implies $f_c \in C^*(\tau) = \Phi^* \subset \Phi$ and, by (c), $f \in \Phi$. In the case of (a), the function $h: R \to R$, $h(u) = |u|$, can be represented as $h(u) = u \vee (-u)$ and also as the infimum of a family of quadratic polynomials; therefore (b) implies that $|f| \in \Phi$ whenever $f \in \Phi$, i.e. that Φ is a lattice. Hence the case of (a), (b), (c) has been reduced to that one of (a'), (b), (c). Finally, in the case of (a), (b), (c'), $f \in \Phi$ implies $g = f/(1 + |f|) \in \Phi^* = C^*(\tau)$, hence $f = g/(1 - |g|) \in C(\tau)$. Conversely $f \in C(\tau)$ implies $g \in C^*(\tau) = \Phi^* \subset \Phi$ and $f \in \Phi$.

Remark 1. By [4], p. 184, (b) implies that Φ is complete; hence (a), (b), (c') $\Rightarrow$ (c) is a consequence of [17], Corollary 3.7, and we get an alternative proof for the case (a), (b), (c').

Another answer to the above question is contained in

Theorem 2. If Φ is a function class on X, the following statements are equivalent:

(a) There are a topological space Y and a surjective map $p: X \to Y$ such that $\Phi = C(Y) \circ p$,

(b) Φ is strongly composition-closed,

(c) There is a topology τ on X such that $\Phi = C(\tau)$.

Proof. (a) $\Rightarrow$ (b): Suppose $f_i \in \Phi$, $f_i = g_i \circ p$, $g_i \in C(Y)$, $i \in I$. Define $h: X \to R^I$ and $g: Y \to R^I$ by $h(x) = (f_i(x))$, $g(y) = (g_i(y))$. Then $h = g \circ p$ and g is continuous, hence if $k \in C(h(X)) = C(g(Y))$, we have $k \circ g \in C(Y)$ and $k \circ h \in C(Y) \circ p$.

(b) $\Rightarrow$ (c) is contained in [5], (2.6), and (c) $\Rightarrow$ (a) is obvious.

A similar argument furnishes, using [14], p. 114 for (b) $\Rightarrow$ (c):

Theorem 3. For a function class on X, the following statements are equivalent:

(a) There are a topological space Y and a map $p: X \to Y$ such that $\Phi = C(Y) \circ p$,

(b) Φ is weakly composition-closed,

(c) There are a set $Y \supset X$ and a topology on Y such that $\Phi = C(Y)|X$.

A modification of this theorem yields:

Theorem 4. For a function class Φ, the following statements are equivalent:

(a) There are a topological space Y and a map $p: X \to Y$ such that $p(X)$ is dense in Y and $\Phi = C(Y) \circ p$,

(b) Φ is composition-closed,

(c) There are a set $Y \supset X$ and a topology on Y such that X is dense in Y and $\Phi = C(Y)|X$.

Proof. (a) $\Rightarrow$ (b): Similarly as in the proof of Theorem 2, by $g(Y) = g(\overline{p(X)}) \subset \overline{g(p(X))} = \overline{h(X)}$.

(b) $\Rightarrow$ (c): Suppose that Φ is cc, set $\Phi = \{f_i : i \in I\}$ with some index set I, and consider h as in (1.1). Choose a set $Y \supset X$ such that there exists a bijection $h': Y - X \to Z - h(X)$ where $Z = \overline{h(X)}$, and define $q: Y \to Z$ by $q(x) = h(x)$ for $x \in X$, $q(x) = h'(x)$ for $x \in Y - X$. Equip Y with the inverse image by q of the topology of Z (i. e. $G \subset Y$ is open iff $G = q^{-1}(H)$ for some $H \subset Z$ open in Z). Now if $f \in \Phi$, say $f = f_i$, $i \in I$, then $f_i = p_i \circ q|X$ where p_i denotes the projection of R^I onto its i-th factor, hence $p_i \circ q \in C(Y)$. Conversely if $g \in C(Y)$, then $x, y \in Y$, $g(x) \neq g(y)$ implies that x and y have disjoint neighbourhoods in Y so that $q(x) \neq q(y)$; therefore $g = k \circ q$ with $k \in C(Z)$ and $g|X = k \circ h \in \Phi$ by the cc property of Φ. Finally X is dense in Y since $h(X) = q(X)$ is dense in $Z = \overline{h(X)}$.

(c) $\Rightarrow$ (a): Obvious.

Remark 2. The proofs of (b) $\Rightarrow$ (c) in Theorems 2 and 3 are quite similar to that one in Theorem 4, only one has to choose $Z = h(X)$ or $Z = R^I$, respectively.

3. Now let X be a given topological space, Φ a function class on X, and let us look for conditions on Φ involving $\Phi = C(X)$ or $\Phi = C(Y)|X$ with a suitable topological space Y containing X as a subspace. Theorems 2, 3, 4 furnish results of this kind provided the spaces in question are completely regular (not necessarily Haus-

dorff).

Theorem 5. Let X be a completely regular space and Φ a function class on X. $\Phi = C(X)$ iff Φ is strongly composition-closed and $Z(\Phi)$ is a closed base in X (i. e. a system of closed sets such that each closed set is the intersection of some elements of this system).

Proof. The necessity follows from Theorem 2 and from the fact that a space is completely regular iff $Z(X)$ is a closed base ([7], p. 38). Conversely if Φ is scc, then, by Theorem 2, it coincides with $C(\tau)$ for a topology τ obtained as the inverse image topology of a subspace of R^I (see Remark 2). Consequently τ is completely regular, thus $Z(\Phi)$ is a closed base for τ, and τ is identical with the given topology of X since both topologies admit the same closed base.

Theorem 6. Let X be a topological space and Φ a function class on X. There is a completely regular space $Y \supset X$ containing X as a subspace and satisfying $\Phi = C(Y)|X$ iff Φ is weakly composition-closed and $Z(\Phi)$ is a closed base in X.

Proof. If $\Phi = C(Y)|X$ and Y is completely regular, then Φ is wcc by Theorem 3 and $Z(\Phi)$, being the trace on X of $Z(Y)$, is a closed base in X. Conversely if Φ is wcc, then $\Phi = C(Y)|X$ by Theorem 3 for a suitable set $Y \supset X$ and a completely regular topology τ on Y (by Remark 2). Hence $Z(Y)$ is a closed base for τ and the same holds for $Z(\Phi)$ and the topology induced by τ on X; consequently the latter coincides with the given topology of X.

A similar argument, using Theorem 4 instead of Theorem 3, furnishes

Theorem 7. Let X be a topological space and Φ a function class on X. There is a completely regular space $Y \supset X$ containing X as a dense subspace and satisfying $\Phi = C(Y)|X$ iff Φ is composition-closed and $Z(\Phi)$ is a closed base in X.

Theorem 8. Let X be a T_0-space and Φ a function class on X. There is a completely regular Hausdorff space $Y \supset X$ containing X as a (dense) subspace iff Φ is weakly composition-closed (composition-closed) and $Z(\Phi)$ is a closed base in X.

Proof. By the proof of Theorem 6 (or 7) the topology on Y is now Hausdorff since the map $q: Y \to Z$ in the proof of Theorem 4 is injective.

Remark 3. The same argument shows that, under the hypotheses of Theorem 8, the space Y can be supposed to be realcompact (see [7], 8.2, 8.10, 8.11; cf. [5], (3.5)).

If the topology given on X is not completely regular, then the condition that $Z(\Phi)$ be a closed base in X is no more necessary for having $\Phi = C(X)$ or $\Phi = C(Y)|X$. However, Theorem 5 can be modified in a suitable way in order to embrace the case of an arbitrary topological space X:

Theorem 9. Let X be a topological space and Φ a function class on X. $\Phi = C(X)$ iff Φ is (countably) strongly composition-closed and $Z(\Phi) = Z(X)$.

Proof. If $\Phi = C(X)$, then Φ is scc (and a fortiori cscc) by Theorem 2 and, of course, $Z(\Phi) = Z(X)$. Conversely if Φ is scc, then $\Phi = C(\tau)$ for a topology τ on X, and $C(\tau)$ is composed of all functions $f: X \to R$ the level sets of which belong to $Z(\tau) = Z(\Phi)$. Similarly $C(X)$ is the class of all functions with level sets in $Z(X)$, hence $Z(\Phi) = Z(X)$ implies $C(\tau) = C(X)$ and $\Phi = C(X)$.

If Φ is cscc, then, by Theorem 10 below, Φ is still composed of all functions the level sets of which belong to $Z(\Phi)$ so that $Z(\Phi) = Z(X)$ implies again $\Phi = C(X)$.

Theorem 10. ([6], Theorem 16). Let Φ be a function class on X. Then the following statements are equivalent:

(a) Φ is countably strongly composition-closed,

(b) Φ is finitely strongly composition-closed and complete,

(c) Φ is finitely composition-closed, inversion-closed, and complete,

(d) Φ is a complete and inversion-closed function ring,

(e) $Z(\Phi)$ is a δ-lattice in X (i. e. $\emptyset, X \in Z(\Phi)$, $Z_1, Z_2 \in Z(\Phi)$ implies $Z_1 \cup Z_2 \in Z(\Phi)$, $Z_i \in Z(\Phi)$ for $i = 1, 2, \ldots$ implies $\bigcap \{Z_i: i = 1, 2, \ldots\} \in Z(\Phi)$), and Φ is composed of all functions the level sets of which belong to $Z(\Phi)$,

(f) There is a δ-lattice in X so that Φ coincides with the class of all functions the level sets of which belong to this δ-lattice.

Remark 4. (a) $\Rightarrow$ (b) follows from [13], Theorem 2.1, (b) $\Rightarrow$ (c) $\Rightarrow$ (d) is obvious, (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) is essentially contained in [11], pp. 236 and 241 (cf. also [17], Theorem B' and [10], Lemma 6.3), finally (d) $\Rightarrow$ (a) is [16], Theorem 4.9.

An interesting consequence of Theorem 10 is the fact that the cscc classes can be characterized as fscc classes having some approximation property (namely completeness), and even the fscc property can be reduced to the condition of being closed under a finite number of suitable operations. Theorems 1 and 2 yield a similar characterization of scc classes.

Problem 1. Is it possible to characterize the cc and wcc classes by the property of being fcc and by suitable approximation properties?

In this direction it is worth-wile to mention that every wcc class is complete ([13], Theorem 2.1).

Problem 2. Let X be an arbitrary topological space and Φ a function class on X. Give necessary and sufficient conditions in order to have $\Phi = C(Y)|X$ for a suitable topological space $Y \supset X$ containing X as a (dense) subspace.

Notice that, in Theorems 6 and 7, complete regularity of Y was essential to assure the necessity of the given conditions.

4. Let now A be an arbitrary ring. We shall study the question under what conditions is A isomorphic with a ring of the form $C(X)$ or $C(Y)|X$. First we examine the (purely algebraic) question when is A isomorphic with a function ring. The following theorem is an easy consequence of well-known results on subdirect sums of rings (see e. g. [15], Theorem 2, or [18]), and is essentially known ([1], [2], [12], [14]):

Theorem 11. For a ring A, there is a function ring isomorphic with A iff the conditions (a), (b), and one of the conditions (c_i) ($i = 1, 2, 3$) below are fulfilled.

(a) A contains a unit element $e \neq 0$,

(b) There is a homomorphism $\chi: R \to A$ such that $e \in \chi(R)$,

(c_1) For every (two-sided) ideal $I \subset A$, $I \neq \{0\}$, there exists a homomorphism $\varphi: I \to R$ such that $1 \in \varphi(I)$,

(c_2) For every ideal $I \subset A$, $I \neq \{0\}$, there is a homomorphism $s\colon A \to R$ such that $s(I) \neq \{0\}$,

(c_3) For every $a \in A$, $a \neq 0$, there is a homomorphism $s\colon A \to R$ such that $s(a) \neq 0$.

Under these hypotheses, a function ring isomorphic with A can be constructed over the set S of all homomorphisms $s\colon A \to R$ such that $s(A) \neq \{0\}$ by assigning to $a \in A$ the function $f\colon S \to R$ defined by $f(s) = s(a)$.

<u>Proof</u>. Assume that Φ is a function ring on X. Then (a), (b), (c_1) hold for Φ instead of A, consequently for any ring A isomorphic with Φ. In fact, the constant 1 is the unit of Φ, the homomorphism χ assigning the constant c to $c \in R$ satisfies (b), and if I is an ideal in Φ, $g \in I$, $g \neq 0$, then there is $x \in X$ with $g(x) \neq 0$, and $\varphi(f) = f(x)$ $(f \in I)$ defines a homomorphism $\varphi\colon I \to R$ satisfying (c_1) since $\varphi(g/c) = 1$ for $c = g(x)$.

For a ring A, (c_1) $\Rightarrow$ (c_2). For, if I is an ideal in A and $\varphi\colon I \to R$ is a homomorphism, $i \in I$, $\varphi(i) = 1$, then $s(a) = \varphi(iai)$ defines a homomorphism $s\colon A \to R$ for which $s(i) = 1$.

(c_2) $\Rightarrow$ (c_3) for any ring A. In fact, if (c_3) were false, then $a \neq 0$ would be contained in the intersection J of the kernels of all homomorphisms $s\colon A \to R$ with $s(A) \neq \{0\}$ (the existence of at least one s of this kind follows from (c_2) for $I = A \neq \{0\}$). By (c_2), there would be a homomorphism $s\colon A \to R$ such that $s(J) \neq \{0\}$ which contradicts to $J \subset \operatorname{Ker} s$.

Finally if A satisfies (a), (b), (c_3), then denote by S the set of all homomorphisms $s\colon A \to R$ such that $s(A) \neq \{0\}$, and define $f_a(s) = s(a)$ for $a \in A$, $s \in S$. Then $S \neq \emptyset$ by (a) and (c_3), and clearly $f_{a+b} = f_a + f_b$, $f_{ab} = f_a f_b$ so that $\Phi = \{f_a\colon a \in A\}$ is a ring under pointwise addition and multiplication, and $\psi(a) = f_a$ defines an epimorphism $\psi\colon A \to \Phi$. Moreover, Φ is a function ring (i. e. it contains all constant functions). In fact, if $s \in S$, then $s \circ \chi\colon R \to R$ is a homomorphism such that $s(\chi(R)) \neq \{0\}$ since $e \in \chi(R)$ by (b) and $s(e) = 0$ would imply $s(a) = s(ae) = s(a)s(e) = 0$ for each $a \in A$. Therefore $s(\chi(c)) = c$ for $c \in R$ ([7], 0.22), i. e. $\psi(\chi(c)) \in \Phi$ is the constant function c. Finally ψ is a monomorphism by (c_3).

Theorem 11, combined with Theorems 2 and 3, furnishes characterizations of those rings that are isomorphic with rings of the form $C(X)$ or $C(Y)|X$. The first one of these problems was investigated by many authors (see e. g. [12],[1],[2],[9]). We shall need two lemmas that are slight modifications of [7], 3.9 and 8.8, and [7], 0.23 and 8.3, respectively.

Lemma 1. Let Y be a topological space and $X \subset Y$. Then there exist a realcompact Tychonoff space Y' and a closed subspace $X' \subset Y'$ such that the rings $C(Y)$ and $C(Y)|X$ are isomorphic with $C(Y')$ and $C(Y')|X'$ respectively.

Proof. Let Y_1 be the set Y equipped with the topology for which $Z(Y)$ is a closed base, and $X_1 = X$. Then $C(Y_1) = C(Y)$, and $C(Y_1)|X_1 = C(Y)|X$. Moreover, Y_1 is completely regular.

Now let Y_2 be the set of all equivalence classes belonging to the equivalence relation introduced in Y_1 by setting $x \sim y$ iff $f(x) = f(y)$ for every $f \in C(Y_1)$. Denote by $q(x)$, for $x \in Y_1$, the equivalence class containing x. Then equip Y_2 with the quotient topology corresponding to $q: Y_1 \to Y_2$, and define $X_2 = q(X_1)$. By assigning $f \circ q$ $(f \circ q|X_1)$ to $f \in C(Y_2)$ $(f|X_2 \in C(Y_2)|X_2)$, we obtain an isomorphism from $C(Y_2)$ onto $C(Y_1)$ (from $C(Y_2)|X_2$ onto $C(Y_1)|X_1$). Moreover, Y_2 is a Tychonoff space.

Finally let Y' denote the Hewitt realcompactification of Y_2 and X' the closure of X_2 in Y'. Then by assigning $f|Y_2$ to $f \in C(Y')$ $(f|X_2$ to $f|X' \in C(Y')|X')$, we get an isomorphism from $C(Y')$ onto $C(Y_2)$ (from $C(Y')|X'$ onto $C(Y_2)|X_2$).

Lemma 2. Let Y be a realcompact Tychonoff space and $X \subset Y$ be closed. Then the ring homomorphisms $s: C(Y)|X \to R$ such that $s(C(Y)|X) \neq \{0\}$ correspond in a one-to-one manner to the points of X, the homomorphism s corresponding to $x \in X$ being defined by $s(f|X) = f(x)$ for $f \in C(Y)$.

Proof. If $f \in C(Y)$, $s(f|X) = c \neq 0$, then $s(rf|X) = rc$ for $r \in R$ so that s is an epimorphism onto R. Denote by $k: C(Y) \to C(Y)|X$ the epimorphism defined by $k(f) = f|X$. Then $s \circ k$ is an epimorphism from $C(Y)$ onto R, hence $\mathrm{Ker}(s \circ k)$ is a real maximal ideal of $C(Y)$, and $s(k(f)) = f(x)$ for some $x \in Y$ depending on s and every $f \in C(Y)$. Clearly $x \in X$ since otherwise there would be an $f \in C(Y)$ vanishing on X but satisfying $f(x) = 1$. Thus $s(f|X) = f(x)$ for some $x \in X$ and every $f \in C(Y)$. Conversely, this equality defines an epimorphism $s: C(Y)|X \to R$, and distinct points x_1 and x_2 generate distinct epimorphisms.

The desired characterization of the rings isomorphic to rings of the form $C(X)$ is now the following:

<u>Theorem 12</u>. A ring A is isomorphic with a ring of the form $C(X)$ iff A satisfies conditions (a), (b), (c_i) $(i = 1, 2, 3)$ of Theorem 11, and the following is true: whenever $a_i \in A$ $(i \in I)$ and S denotes the set of all homomorphisms $s: A \to R$ such that $s(A) \neq \{0\}$, further $h: S \to R^I$ is defined by $h(s) = (s(a_i))$, finally $k \in C(h(S))$, then there is an $a \in A$ such that $k(h(s)) = s(a)$ for every $s \in S$.

<u>Proof</u>. If these conditions are fulfilled, then, by Theorem 11, an isomorphism $\psi: A \to \Phi$ is given by $\psi(a) = f$ where f is a function defined on S such that $f(s) = s(a)$ for $s \in S$, and $\Phi = \psi(A)$. Now our hypotheses assure precisely that Φ is scc so that, by Theorem 2, $\Phi = C(\tau)$ for a suitable topology on S.

Conversely let $\omega: C(X) \to A$ be an isomorphism, X a topological space. By Lemma 1 we can suppose that X is realcompact. Then, by Theorem 11, A fulfils (a), (b), (c_i). Moreover, $\{s \circ \omega: s \in S\}$ is the set of all non-vanishing homomorphisms from $C(X)$ into R, hence, by Lemma 2, we obtain a one-to-one correspondence between the homomorphisms $s \circ \omega$ and the points $x \in X$ by putting $s(\omega(f)) = f(x)$. By Theorem 2, $C(X)$ is scc, and this yields the last condition for A.

A similar argument furnishes, by using Theorem 3 instead of Theorem 2, and taking into account that the wcc and cwcc properties coincide:

<u>Theorem 13</u>. A ring A is isomorphic with a ring of the form $C(Y)|X$ iff A satisfies conditions (a), (b), (c_i) $(i = 1, 2, \text{or } 3)$ of Theorem 11, and the following is true: whenever $a_i \in A$ $(i = 1, 2, 3, \ldots)$ and S is the same as in Theorem 11, further $h: S \to R^N$ is defined by $h(s) = (s(a_i))$, finally $k \in C(R^N)$, then there is an $a \in A$ such that $k(h(s)) = s(a)$ for every $s \in S$.

<u>5</u>. Finally we investigate the following problem. Given a function ring Φ on a set T, look for conditions assuring that Φ be isomorphic with a ring of the form $C(X)$ or $C(Y)|X$. For this purpose, we need a slight modification of [7], 10.6:

<u>Lemma 3</u>. Let Φ be a function ring on a set T and Z a realcompact Tychonoff space. If $\omega: C(Z) \to \Phi$ is an epimorphism, then there is a map $p: T \to Z$ such that $\omega(g) = g \circ p$ for $g \in C(Z)$. If ω is an isomorphism, then $p(T)$ is dense in Z.

<u>Proof</u>. For $t \in T$, $s(g) = \omega(g)(t)$ defines an epimorphism s:

$C(Z) \to R$. Hence there is a unique $z \in Z$ satisfying $\omega(g)(t) = g(z)$. Define $z = p(t)$; then clearly $\omega(g) = g \circ p$. If $p(T)$ is not dense in Z, then there are $g_1, g_2 \in C(Z)$, $g_1 \neq g_2$ with $g_1 \circ p = g_2 \circ p$ and ω cannot be an isomorphism.

Theorem 14. Let Φ be a function ring on a set T. There is a ring $C(X)$ isomorphic with Φ iff Φ is composition-closed.

Proof. If Φ is cc, then, by Theorem 4, there is a topological space X containing T as a dense subset and satisfying $\Phi = C(X)|T$. Then clearly $C(X)$ is isomorphic with Φ, an isomorphism being obtained by assigning $f|T$ to $f \in C(X)$.

Conversely suppose that there is an isomorphism $\omega: C(Z) \to \Phi$. By Lemma 1, we can suppose that Z is a realcompact Tychonoff space. Consider the map $p: T \to Z$ of Lemma 3, and choose a set $X \supset T$ such that there is a bijection $p': X - T \to Z - p(T)$. Define $q: X \to Z$ by $q(x) = p(x)$ for $x \in T$, $q(x) = p'(x)$ for $x \in X - T$. Equip X with the inverse image by q of the topology of Z. Then T is dense in X, and the elements of $C(X)$ are precisely the functions $g \circ q$ where $g \in C(Z)$. Clearly $\omega(g)(t) = g(p(t))$ for $g \in C(Z)$, $t \in T$, hence $\omega(g) = g \circ q|T$ and $\Phi = C(X)|T$ so that Φ is cc by Theorem 4.

Theorem 15. A function ring Φ on a set T is isomorphic with a ring of the form $C(Y)|X$ iff Φ is weakly composition-closed.

Proof. If Φ is wcc, then Theorem 3 yields $\Phi = C(Y)|T$ for some topological space $Y \supset T$. Conversely, if $\chi: C(Z)|Y \to \Phi$ is an isomorphism, then, by Lemma 1, Z can be supposed to be realcompact and Tychonoff. Define $\varphi: C(Z) \to C(Z)|Y$ by $\varphi(g) = g|Y$ and apply Lemma 3 for the epimorphism $\omega = \chi \circ \varphi: C(Z) \to \Phi$. Define p, X, p', q and the topology on X as in the proof of Theorem 14. Then Φ is composed of the functions $\omega(g)$ and $C(X)$ of those $g \circ q$ for $g \in C(Z)$, moreover $\omega(g) = g \circ q|T$. Hence $\Phi = C(X)|T$, and Φ is wcc by Theorem 3.

Remark 5. Theorem 15 can be obtained with the help of some results on uniform spaces ([8], [3]).

Remark 6. The function ring composed of all polynomials on the real line is not wcc (in fact, it is not complete), hence this ring is not isomorphic with any ring $C(Y)|X$. On the other hand, if Φ is the first Baire class on R, then Φ is wcc (in fact, it is cscc by Theorem 10) without being cc ([5], (2.5)), hence Φ is of the form

$C(Y)|R$ for a suitable space $Y \supset R$ without being isomorphic with a ring $C(X)$. A function ring Φ on T that is cc without being scc (cf. [5], p. 147) is isomorphic with a ring $C(X)$ but fails to be of the form $C(T)$.

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