Arthur Harold Stone
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MEASURE-PRESERVING MAPS

A. H. Stone

Many important spaces come equipped with measures as well as with topologies. Thus it is of significance to investigate maps that are measure-preserving as well as continuous; and some problems of this nature will be considered here. By a "topological measure space" $(X, \mathcal{T}, \mu)$, or (X, μ) for short, we mean a set X with a topology $\mathcal{T}$ and a countably additive, non-negative regular Borel measure μ, completed with respect to null sets. A map $f: (X, \mathcal{T}, \mu) \to (Y, \mathcal{U}, \nu)$ is "measure-preserving" provided that, for all ν-measurable subsets B of Y, $f^{-1}(B)$ is μ-measurable and $\mu(f^{-1}(B)) = \nu(B)$.

One striking example of a measure-preserving map is Peano's well-known continuous map, say ϕ, of the unit interval I onto the unit square I^2. It is perhaps not well-known that ϕ is measure-preserving when I and I^2 have their usual Lebesgue measures; but this is easily seen by noting that, at the n^{th} stage of the construction of ϕ, I^2 is subdivided into $2^n \times 2^n$ equal squares, the inverse of each of which is an interval in I of the right length (4^{-n}). Recently Schoenfeld [3] has generalized this observation into a measure-preserving form of the Hahn-Mazurkiewicz theorem: if X is a Peano space, with a measure μ such that $\mu(X) = 1$ and μ is positive for every non-empty open subset of X, then there is a continuous measure-preserving map of I (with Lebesgue measure λ) onto X. (Conversely, the conditions on μ here are obviously necessary.)

One of the first major theorems about measure-preserving maps is that of Oxtoby and Ulam [2, p. 886] (see also [5]): if m is a measure on Euclidean space R^n that is non-atomic, positive for all non-empty open sets, and σ-finite but not finite, then there is a measure-preserving homeomorphism from (R^n, m) onto (R^n, λ_n), where λ_n denotes n-dimensional Lebesgue measure. This theorem is deduced from an analogous characterization of Lebesgue measure on I^n : there is a measure-preserving homeomorphism of (I^n, m) onto (I^n, λ_n) if (and only if) m is non-atomic, positive for non-empty open sets, vanishes on the boundary of I^n, and $m(I^n) = 1$. It would be highly desirable to have an extension of this theorem to characterize the product Lebesgue measure λ_∞ on the Hilbert cube I^∞ ; but this seems to be difficult.

A natural question here is: What spaces (X, μ) can be embedded in $(I^\infty, \lambda_\infty)$ by measure-preserving homeomorphisms ? Of course, X must be separable and metrizable, μ must be non-atomic, and $\mu(X)$ must be ≤ 1. The case $\mu(X) = 1$ would imply the extension of the Oxtoby-Ulam theorem just mentioned, so we assume $\mu(X) < 1$. In this form, the question was raised in [4], and answered only in very special cases. The answer is still unknown in general, but the following theorem provides a partial answer that improves on the results stated in [4].

Theorem 1 *If X is a finite-dimensional separable metric space, with a non-atomic complete regular Borel measure μ for which $\mu(X) < 1$, then there is a measure-preserving homeomorphism of (X, μ) onto a subspace of $(I^\infty, \lambda_\infty)$.*

Proof Let $\dim X = n$. Take J to be the interval $[0, 1-\varepsilon] \subset I$, where ε is positive and small enough so $(1-\varepsilon)^{2n+2} > \mu(X)$. First we establish the theorem with $(I^\infty, \lambda_\infty)$ replaced by $(J^{2n+2}, \lambda_{2n+2})$. Take a closed interval K interior to J , and consider $Z = K^{2n+1} \times \{\frac{1}{2}\} \subset J^{2n+2}$. There is a homeomorphism f of X onto a subset Y of Z (see [1, p. 60]). Define a Borel measure m on J^{2n+2} by: $m(B) = \mu(f^{-1}(B \cap Y)) + ((1-\varepsilon)^{2n+2} - \mu(X))\lambda^*_{2n+2}(B - Y)$, where λ^*_{2n+2} is outer Lebesgue measure. Then m (completed with respect to null sets) satisfies the hypotheses of the Oxtoby-Ulam theorem, so that there exists a measure-preserving homeomorphism g of (J^{2n+2}, m) onto $(J^{2n+2}, \lambda_{2n+2})$. Now $g \circ f$ is the measure-preserving embedding of (X, μ) into $(J^{2n+2}, \lambda_{2n+2})$, as required.

We remark, parenthetically, that if X is compact we can replace $2n+2$ by $2n+1$ here, by taking $Y \subset K^{2n+1} \subset J^{2n+1}$. Since Y is closed and n-dimensional it is automatically nowhere dense, so m will still be positive on non-empty open sets.

Next we observe that the interval (J, λ) can be embedded, by a measure-preserving homeomorphism, into $(I^\infty, \lambda_\infty)$. The construction is roughly as follows. It is not hard to see that one can construct a Cantor set C_1 in I^∞ such that $\lambda_\infty(C_1) > 1 - \varepsilon$, and that one can run a simple arc A_1 through C_1. Near each of the countably many complementary intervals of $A_1 - C_1$, we place a small Cantor set of positive measure, and modify A_1 to run through it. Iterating, we end with a simple arc $A \subset I^\infty$ such that $\lambda_\infty(A) > 1 - \varepsilon$ and each sub-arc of A has positive λ_∞-measure. Take a sub-arc A^* of A having $\lambda_\infty(A^*) = 1 - \varepsilon$, and take a homeomorphism h of J onto A^*. For each $t \in J$ put $\phi(t) = \lambda_\infty(h[0, t])$. Then ϕ is a continuous and strictly increasing function, hence a homeomorphism of J onto J; and $h \circ \phi^{-1}$ is a measure-preserving homeomorphism of (J, λ) into $(I^\infty, \lambda_\infty)$, as required.

It follows at once that $(J^{2n+2}, \lambda_{2n+2})$ is imbeddable, by a measure-preserving homeomorphism ψ, into the (topological and measuretheoretic) product $(I^\infty, \lambda_\infty)^{2n+2}$. But this is just $(I^\infty, \lambda_\infty)$. Thus $\Theta = \psi \circ g \circ f$ gives a measure-preserving homeomorphism of (X, μ) into $(I^\infty, \lambda_\infty)$, as required.

<u>Remark</u> It cannot be asserted (without extra hypotheses -- for instance, that X is compact, or even analytic) that the images of X in I^{2n+2}, or in I^∞, are Lebesgue measurable. For a measurable subset of positive Lebesgue measure in I^n $(n \leq \infty)$ must contain a Cantor set of positive measure; and this need not be true of (X, μ). Of course, Lebesgue outer measure induces a completed Borel measure on the image of X, whether or not it is measurable; and this is the measure that is "preserved" in Theorem 1.

In a different direction, we have a measure-preserving analogue of Urysohn's Lemma :

<u>Theorem</u> 2 <u>Let</u> X <u>be a topological space</u>, <u>and</u> μ <u>a non-atomic Baire measure on</u> X <u>such that</u> $\mu(X) = 1$. <u>Let</u> F_0, F_1 <u>be disjoint zero-sets in</u> X, <u>both of</u> μ-<u>measure</u> 0. <u>Then there exists a continuous measure-preserving map</u> $g:(X, \mu) \to (I, \lambda)$ <u>such that</u> $g^{-1}(0) \supset F_0$ <u>and</u> $g^{-1}(1) \supset F_1$.

<u>Proof</u> Let $\mathfrak{B}$ denote the family of Baire subsets of X. We first note two well-known (and easily proved) facts, the first of which is a consequence of the fact that μ is non-atomic.

(1) If $A \in \mathfrak{B}$ and $\mu(A) = \alpha > \beta > 0$, there exists $B \in \mathfrak{B}$ such that $B \subset A$ and $\mu(B) = \beta$.

(2) Given a zero-set F contained in a cozero-set G, there exists a continuous function $f:X \to I$ such that $F = f^{-1}(0)$ and $X - G = f^{-1}(1)$.

We deduce:

(3) Given a zero-set F contained in a cozero-set G, and $\varepsilon > 0$, there exist a cozero-set U and a zero-set $\tilde{U}$ such that $F \subset U \subset \tilde{U} \subset G$, $\mu(\tilde{U}) < \mu(F) + \varepsilon$, and $\mu(\tilde{U} - U) = 0$.

To prove this, apply (2) and consider the function $g:I \to I$ where $g(t) = \mu(f^{-1}[0, t])$. Then g is a non-decreasing function, hence continuous except for at most countably many values of t. Also g is continuous on the right, hence continuous at 0. Thus there exists $\delta > 0$ such that $0 \leq t < \delta \Rightarrow g(t) < \mu(F) + \varepsilon$. Choose $t_0 \in (0, \delta]$ to be a point of continuity of g, and take $U = f^{-1}[0, t)$, $\tilde{U} = f^{-1}[0, t]$.

In what follows, we continue the same notation: U denotes a cozero-set, and $\tilde{U}$ denotes a zero-set containing U (and hence $\bar{U}$) such that $\mu(\tilde{U} - U) = 0$.

(4) Given a zero-set F contained in a cozero-set G, where $\mu(G - F) = \alpha > 0$, there exist U, $\tilde{U}$ such that $F \subset U \subset \tilde{U} \subset G$ and $\mu(U - F)$ (and consequently also $\mu(G - \tilde{U})$ is between $\alpha/3$ and $2\alpha/3$.

Proof: Take $\varepsilon = \alpha/12$, and apply (3) to get $F \subset U_0 \subset \tilde{U}_0 \subset G$ with $\mu(U_0 - F) < \varepsilon$, and therefore $\alpha - \varepsilon < \mu(G - \tilde{U}_0) \leq \alpha$. From (1), $G - \tilde{U}_0$ contains a Baire set B such that $\mu(B) = \alpha/2$. Since μ (as a finite Baire measure) is automatically regular, there exists a zero-set $Z \subset B$ such that $\mu(B - Z) < \varepsilon$; thus $\alpha/2 - \varepsilon < \mu(Z) \leq \alpha/2$. Applying (3) to Z and $G - \tilde{U}_0$, we get U_1 and $\tilde{U}_1$ such that $Z \subset U_1 \subset \tilde{U}_1 \subset G - \tilde{U}_0$ and $\mu(\tilde{U}_1) < \varepsilon + \mu(Z)$. Put $U = U_0 \cup U_1$, $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$; it is easy to verify that the requirements are satisfied.

Now, under the hypotheses of Theorem 2, write $G(0) = \emptyset$, $F(0) = F_0$, $G(1) = X - F_1$, $F(1) = X$. Applying (4) to $F(0)$ and $G(1)$, we get a cozero-set $G(1/2)$ and a zero-set $F(1/2)$ such that $\mu(F(1/2) - G(1/2)) = 0$, $F(0) \subset G(1/2) \subset F(1/2) \subset G(1)$, and both $\mu(G(1) - F(1/2))$ and $\mu(G(1/2) - F(0))$ are between $1/3$ and $2/3$.

Just as in the proof of the classical Urysohn Lemma, we iterate this procedure, obtaining a system of sets $F(\rho)$, $G(\rho)$, defined for all binary rational numbers ρ in $[0, 1]$, with the following properties.

(In what follows, it is understood that ρ, σ denote binary rationals in $[0, 1]$.) Then $F(\rho)$ is a zero-set, $G(\rho)$ is a cozero-set, $G(\rho) \subset F(\rho) \subset G(\sigma) \subset F(\sigma)$ whenever $\rho < \sigma$, and $\mu(F(\rho) - G(\rho)) = 0$. Further, in inserting the sets $G((2p+1)/2^{q+1})$ and $F((2p+1)/2^{q+1})$ between $F(p/2^q)$ and $G((p+1)/2^q)$ at the $(q+1)^{st}$ stage, we arrange that both $\mu(G((2p+1)/2^{q+1}) - F(p/2^q))$ and $\mu(G((p+1)/2^q) - F((2p+1)/2^{q+1})$ are less than $(2/3)^{q+1}$, and both are greater than $(1/3)^{q+1}$.

Now define $f(x) = \sup\{\rho \mid x \notin F(\rho)\}$ for $x \in G(1) - F(0)$. A straightforward verification shows that $f(x) = \inf\{\sigma \mid x \in G(\sigma)\}$. Further, if we define $f(x) = 0$ for $x \in F(0)$, and $f(x) = 1$ for $x \in X - G(1)$, then $f: X \to I$ is continuous, and if $\rho < t < \sigma$, then $f^{-1}(t) \subset G(\sigma) - F(\rho)$. It follows that $\mu(f^{-1}(t)) = 0$ for all $t \in I$, and thus that $\mu(f^{-1}[0, t]) = \mu(f^{-1}[0, t))$.

Finally, define $\phi: I \to I$ by $\phi(t) = \mu(f^{-1}[0, t])$. It follows from the construction that ϕ is a strictly increasing continuous function, and thence that ϕ is a homeomorphism of I onto I. Put $g = \phi \circ f$; it is easy to see that g fulfils all the requirements of the theorem.

Corollary *Let $(X, \mathcal{T}, \mu)$ be a topological measure space such that $(X, \mathcal{T})$ is normal, μ is non-atomic, and $\mu(X) = 1$. Let F_0, F_1 be disjoint closed sets in X, both of measure 0. Then there exists a continuous measure-preserving map $g: (X, \mu) \to (I, \lambda)$ such that $g^{-1}(0) \supset F_0$ and $g^{-1}(1) \supset F_1$.*

To deduce the Corollary from the theorem, it is enough to show that F_0, F_1, are contained in disjoint zero-sets of measure 0. The regularity of μ gives, for $n = 1, 2, \ldots$, an open set U_n such that $F_0 \subset U_n \subset X - F_1$ and $\mu(U_n) < 1/n$. From the classical Urysohn lemma, there is a continuous function separating F_0 and $X - U_n$, from which we get a zero-set H_n such that $F_0 \subset H_n \subset U_n$. Then $F_0^* = \bigcap_{n=1}^{\infty} H_n$ is a zero-set of measure 0 containing F_0 and disjoint from F_1. Repetition of the argument gives a null zero-set $F_1^* \supset F_1$ and disjoint from F_0^*, as required.

Remark In Theorem 2 (and its Corollary) it would be interesting to know whether one can further arrange that $g(X)$ is a measurable subset of I. The construction used does not in fact ensure this (unless, for instance, X is compact or analytic).

Obviously one could attach to any theorem about the existence of continuous maps the requirement that the maps be measure-preserving, and investigate whether the theorem remains true. We have seen that this is the case (under some restrictions) for Urysohn's Lemma and Urysohn's imbedding theorem. But the "natural" analogue of Tietze's extension theorem is false. For instance, consider the case $(X,\mu) = (I,\lambda)$, $A = [0, 1/2]$, and let $f:A \to I$ be defined by $f(x) = 1/2 - x$ (for $0 \leq x \leq 1/2$). Then f is a continuous measure-preserving map of A onto $[0, 1/2]$; but it has no extension to a continuous measure-preserving map $f^*:X \to I$. For the continuity of f^* at $x = 1/2$ would give $\varepsilon > 0$ such that $f^*[1/2, 1/2 + \varepsilon] \subset [0, 1/2)$, and then $f^{*-1}[0, 1/2) \supset [0, 1/2 + \varepsilon]$ and thus has measure $> 1/2$. It would be interesting to have a satisfactory measure-preserving analogue here.

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University of Rochester,
Rochester, N.Y. 14627, U.S.A.