

Jan Pavelka  
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## ON CLOSED LATTICES

by

Jan PAVELKA

Let  $T$  be a set. Consider the contravariant hom-functor

$$T^{(-)}: \text{Set} \longrightarrow \text{Set}.$$

For an ordinal number  $\alpha$  there is the natural isomorphism

$$\mathcal{K}_X(T^X)^\alpha \approx (T^\alpha)^X$$

defined

$$[\mathcal{K}_X(f_\xi \mid \xi < \alpha)] x = (f_\xi x \mid \xi < \alpha)$$

If we are now given an  $\alpha$ -ary operation  $\sigma: T^\alpha \rightarrow T$  on  $T$  there is a natural way to define operations  $\sigma_X$  on the sets  $T^X$ ; we put

$$\sigma_X = \sigma^X \cdot \mathcal{K}_X.$$

In this context the word "natural" has a precise meaning, namely: for any map  $f: X \rightarrow Y$  the map  $T^f: T^Y \rightarrow T^X$  carries a homomorphism between the algebras  $\langle T^Y, \sigma_Y \rangle, \langle T^X, \sigma_X \rangle$ ; in other words, the maps  $\sigma_X$  form a natural transformation

$$\sigma: (T^{(-)})^\alpha \longrightarrow T^{(-)}. \quad (0.1)$$

Moreover, by Yoneda lemma any transformation (0.1) is induced in this way by the operation

$$\sigma_{T^\alpha}(p_\xi \mid \xi < \alpha)$$

on  $T$ , where  $p_\xi: T^\alpha \rightarrow T$  are projections from the cartesian power of  $T$ .

It is in the above sense that the power set  $PX \approx 2^X$

of a set  $X$  owes its natural structure of a complete boolean algebra to the two-point boolean algebra  $\mathbb{2}$ . Thus, if we want to follow the idea of Zadeh and replace  $\mathbb{2}$  by another set  $T$  of membership degrees to obtain "T-fuzzy sets", then the crucial problem is to devise an algebraic structure on which would yield the most fruitful generalization of the set-theoretical calculus.

As the present state of affairs indicates, the structure of a closed lattice, which is the topic of this introductory article, is one of the most likely candidates.

### 1. Closedness structures on lattices

The general theory of closed categories as developed in [2] deals with the following situation: on a category  $\mathcal{K}$  bifunctors

$$\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}, \quad [-, -] : \mathcal{K}^* \times \mathcal{K} \rightarrow \mathcal{K} \quad (1.1)$$

are given, together with an adjunction in two variables

$$\sigma_{XYZ} : \mathcal{K}(X \otimes Y, Z) \approx \mathcal{K}(X, [YZ]). \quad (1.2)$$

Here we shall be concerned with the special case that arises when  $\mathcal{K}$  is small, thin, and skeletal; i.e. a partially ordered set.

For a partially ordered set  $P$ , a couple of adjoint bifunctors (1.1) reduces to a couple of binary operations

such that

$$(1.3) \quad \otimes : P \times P \rightarrow P \text{ is isotone in both variables,}$$

$$(1.4) \quad [-, -] : P \times P \rightarrow P \text{ is isotone in the second and antitone in the first variable,}$$

(1.5) the adjointness condition

$$x \otimes y \leq z \text{ iff } x \leq [yz]$$

holds for all  $x, y, z \in P$ .

Alternatively, (1.5) can be expressed as

(1.6) for all  $x, y \in P$ ,  $x \leq [y, x \otimes y]$  and  $[xy] \otimes x \leq y$ .

The strength that (1.5) possesses as a link between  $\otimes$  and  $[-, -]$  is perhaps best illustrated by the following

1.1. Proposition (cf.[6]). Let  $\otimes, [-, -]$  be as above and let  $e \in P$ . Then the following are couples of equivalent conditions on  $P$ :

$$(T1) \quad (x \otimes y) \otimes z \leq x \otimes (y \otimes z) \quad [yz] \leq [[xy] [xz]] \quad (H1)$$

$$(T2) \quad x \otimes e = x \quad x = [ex] \quad (H2)$$

$$(T3) \quad e \otimes x = x \quad x \leq y \text{ iff } e \leq [xy] \quad (H3)$$

$$(T4) \quad x \otimes y = y \otimes x \quad x \leq [yz] \text{ iff } y \leq [xz] \quad (H4)$$

$$(T5) \quad (x \otimes y) \otimes z = x \otimes (y \otimes z) \quad [x \otimes y, z] = [x [yz]] \quad (H5)$$

In accordance with [2] we shall use the term "monoidal closed (shortly, MC) structure on  $P$ " to designate a triple  $\langle \otimes, [-, -], e \rangle$  where  $\otimes, [-, -]$  satisfy (1.3) - (1.5) and  $\otimes, e$  satisfy (T2), (T3), and (T5). If, moreover,  $\otimes$  satisfies (T4), we shall speak about a symmetric monoidal closed (shortly SMC) structure on  $P$ .

Observe that

- (i) (T1) together with (T4) already imply (T5) so that an SMC-structure is also defined by axioms (T1) - (T4).
- (ii) (T4) together with (H5) imply

$$(H4!) \quad [x[yz]] = [y[xz]] ,$$

which therefore holds in any SMC-structure.

(iii) For a general MC-structure the condition (H5), in view of (H3), implies (1.5) and can be understood to express the adjointness "inside P".

(iv) (H4) says that the antitone functions  $[-, \times]$  are right self-adjoints. A remark analogous to (iii) can be made about (H4!).

For any couple of adjoint operations  $\langle \otimes, [-, -] \rangle$  we also have

(T0) the functions  $- \otimes x: P \rightarrow P: x \in P$  preserve all suprema that exist in P, and

(H0) the functions  $[x, -]: P \rightarrow P: x \in P$  preserve all infima that exist in P.

Moreover, if P is a complete lattice (as we shall assume from now on) then

(i) any isotone binary operation  $\otimes$  satisfying (T0) has a right adjoint, uniquely determined by  $\otimes$ , whose values are given by the formula

$$[yz] = \bigvee \{x \mid x \otimes y \leq z\} \quad (1.7)$$

(ii) Any antiisotone-isotone binary operation  $[-, -]$  satisfying (H0) has a left adjoint, uniquely determined by  $[-, -]$ , whose values are given by the formula

$$x \otimes y = \bigwedge \{z \mid x \leq [yz]\} \quad (1.8)$$

Thus any isotone associative operation  $\otimes$  with a two-sided unit e on a complete lattice L is a part of an MC-structure on L iff it satisfies (T0). In [1] a complete lattice

endowed with such an operation is called a cl-monoid provided it also satisfies

(TO)\* the functions  $x \otimes - : L \rightarrow L$ ;  $x \in L$  preserve all suprema.

Remark: Unless  $\otimes$  is commutative, (TO)\* does not follow from (TO). What can be proved is that it is equivalent to

(HO)\* the functions  $[-, x] : L \rightarrow L$ ;  $x \in L$  transfer all suprema to infima.

Indeed, assume (TO)\* and let  $Y \subseteq L$ ,  $z \in L$ . Since  $[-, z]$  is antitone we have  $[\bigvee Y, z] \leq \bigwedge [Y, z]$ . Hence it suffices to show that the converse inequality also holds in order to prove (HO)\*. By (1.5),  $\bigwedge [Y, z] \leq [\bigvee Y, z]$  iff  $\bigwedge [Y, z] \otimes \bigvee Y \leq z$ . By (TO),  $\bigwedge [Y, z] \otimes \bigvee Y = \bigvee \{ \bigwedge [Y, z] \otimes y \mid y \in Y \}$ . Because  $\bigwedge [Y, z] \otimes y \leq [yz] \otimes y \leq z$  holds for any  $y \in Y$  this part is complete. The proof of  $(HO)^* \implies (TO)^*$  is analogous.

Still, it is useful to assume (TO) for non-commutative  $\otimes$  - it guarantees that the opposite multiplication

$$x \otimes^* y = y \otimes x$$

also has an adjoint, say  $[-, -]^*$ , and allows us to treat both resulting MC-structures simultaneously, as it is done in [4]. Trivially one then has

$$x \leq [yz] \text{ iff } y \leq [xz]^*$$

so that each  $[-, z]$  is again a contravariant adjoint on the right. If  $\otimes$  is commutative then  $[-, -]$  and  $[-, -]^*$  coincide and  $[-, z]$  is self-adjoint, as was already noted.

1.2. Example: If a complete lattice  $L$  is completely distributive then  $\langle L, \wedge, 1 \rangle$ , where  $\wedge$  is the meet in  $L$  and  $1$  is the greatest element of  $L$ , is a commutative cl-monoid and therefore a part of an SMC-structure on  $L$ : the so called Heyting algebra on  $L$ . The right adjoint to  $\wedge$  is denoted by  $y \rightarrow z$  and called the Heyting operation on  $L$ . (Concerning terminology: a category  $\mathcal{K}$  is called cartesian closed if it has finite products and the product functor is a left adjoint. Since  $x \wedge y$  is the categorial product of the objects  $x, y \in L$ , we have completely distributive = cartesian closed.)

1.3. Example: Let  $R$  be an associative ring with unit. Let  $L$  denote the complete lattice of all two-sided ideals in  $R$ . The operation

$$\mathcal{X} \otimes \mathcal{Y} = \left\{ \sum_{i=1}^n a_i b_i \mid n \in \mathbb{N}, a_i \in \mathcal{X}, b_i \in \mathcal{Y} \right\}$$

makes  $L$  into a cl-monoid. Again, the unit coincides with the greatest element  $1 = R$  of  $L$ . If  $R$  is commutative, so is  $\otimes$ .

In the sequel we shall restrict our attention to those MC-structures on a lattice  $L$  whose unit coincides with the unit of  $L$  (for the reason see [7]) and call them, in accordance with [1], integral MC-structures.

The latter example, though rather esoteric from our point of view, motivated an investigation of integral MC-structures on lattices carried out - long before the concept of a closed category was proposed - by R.P. Dilworth and M. Ward (see [3],[4]). Concerning terminology:

in [3], the operation  $[-, -]$  is called residuation

and  $[x, y]$  is denoted by  $y:x$ .

In [4] where non-symmetric structures are considered,

$\left\{ \begin{array}{l} [xy] \\ [xy]^* \end{array} \right\}$  is denoted by  $\left\{ \begin{array}{l} y \cdot x^{-1} \\ x^{-1} \cdot y \end{array} \right\}$  and called

the  $\left\{ \begin{array}{l} \text{left} \\ \text{right} \end{array} \right\}$  residual of  $x$  with respect to  $y$ .

The above mentioned papers concentrate on the following questions.

$Q_1$ : Do lattices with nice properties admit integral MC-structures?

$Q_2$ : How far does an integral MC-structure on  $L$  affect its lattice structure?

We shall not go into much detail here, but let us at least touch  $Q_1$ .

As to completely distributive lattices  $Q_1$  is settled in the affirmative (Example 1.2). Anyway, complete distributivity is not a necessary condition because we have

1.4. Example: Let  $L$  be a complete lattice containing an element  $a$  with the properties

- (i)  $a < 1$ ,
- (ii) for any  $x \in L$  either  $x \leq a$  or  $x \geq a$ ,
- (iii) the sublattice  $\{x \mid x \geq a\}$  is a chain.

Put

$$x \otimes y = \begin{cases} 0 & \text{if} \\ x \wedge y & \text{otherwise.} \end{cases}$$

Then it is easy to check that  $\otimes$  makes  $L$  into a commuta-



tive integral cl-monoid.

The above example suggests that the answer to  $Q_1$  depends primarily on the behavior of  $L$  "in the vicinity of 1". Indeed, let  $L$  admit an integral MC-structure and let  $\otimes$  be its multiplication. Then by monotony of  $\otimes$

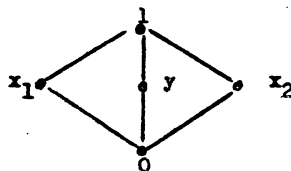
$$(1.9) \quad x \otimes y \leq (1 \otimes y) \wedge (x \otimes 1) = x \wedge y$$

and by (T0)

$$(1.10) \quad \text{for any } X \subseteq L, y \in L: \bigvee X = 1 \implies$$

$$\implies y = (\bigvee X) \otimes y = \bigvee \{x \otimes y \mid x \in X\} \leq \bigvee \{x \wedge y \mid x \in X\}.$$

Now observe that this already disqualifies the modular lattice



( $X = \{x_1, x_2\}$  and  $y$  fail to satisfy the necessary condition (1.10)).

## 2. Special lattices: chains and boolean algebras

We have already noted that if the lattice  $L$  in question is completely distributive, questions  $Q_1$  and  $Q_2$  cease to be interesting. On the other hand, new problems arise such as:  
 $Q_3$ : How many integral MC (or, in particular, SMC) structures does  $L$  admit? Can they be described and classified in a reasonable way?

$Q_4$ : For two integral MC-structures  $\mathcal{G}_1 = \langle \otimes_1, [-, -]_1 \rangle$ ,

$\mathcal{G}_2 = \langle \otimes_2, [-, -]_2 \rangle$  on  $L$  put

$$\mathcal{G}_1 \leq \mathcal{G}_2 \equiv_{\text{def}} x \otimes_1 y \leq x \otimes_2 y \text{ for all } x, y \in L$$

(this condition is equivalent to  $[xy]_1 \geq [xy]_2$ , all  $x, y \in L$ ). (1.9) says that the Heyting algebra on  $L$  is the greatest element of the resulting partially ordered set

$\mathcal{G}(L)$ . What else can be proved about  $\mathcal{G}(L)$ ?

$Q_5$ : Given an SMC-structure on  $L$ , the function  $\varphi = [-, 0]$  has the following properties:

- (i)  $\varphi$  sends all suprema to infima,
- (ii)  $\varphi x \geq x \rightarrow 0$  for all  $x \in L$ ,
- (iii)  $x \leq \varphi \varphi x$  for all  $x \in L$ .

Given a function  $\varphi : L \rightarrow L$  satisfying (i) - (iii), does there exist an SMC-structure on  $L$  with  $\varphi = [-, 0]$ ?

We shall end this article by several results in this direction.

2.1. Proposition ([3]). Every boolean algebra admits exactly one MC-structure.

Proof. First we realize that a completely distributive lattice is a boolean algebra iff  $x \vee (x \rightarrow 0) = 1$  holds for all  $x \in L$ . Now suppose  $\langle \otimes, [-, -] \rangle$  is an integral MC-structure on  $L$ . For any  $x \in L$  we have  $x = (x \vee (x \rightarrow 0)) \otimes x = (x \otimes x) \vee ((x \rightarrow 0) \otimes x) \leq (x \otimes x) \vee ([x, 0] \otimes x) = x \otimes x$  whence  $x \wedge y \leq (x \wedge y) \otimes (x \wedge y) \leq x \otimes y$  holds for all  $x, y \in L$ . By (1.9), the proof is complete.

On the other hand, there are  $2^{2^{\aleph_0}}$  non-isomorphic SMC-structures on the compact interval of reals (see [5]) with

continuous multiplication. To these, a theorem of Mostert and Shields can be applied to obtain a fairly simple description of all possibilities. The classification of general SCM-structures on  $I = [0, 1]$  seems to be an open problem.

2.2. Proposition. Denote by  $\mathcal{S}_0(L)$  the subposet of  $\mathcal{S}(L)$  consisting of all symmetric elements. Every nonempty chain in  $\mathcal{S}_0(L)$  has a supremum.

Proof. Let  $\beta \neq \alpha$  be a chain in  $\mathcal{S}_0(L)$ . We show that

$$x \overline{\otimes} y = \bigvee \{x \otimes y \mid \langle \otimes, [-, -] \rangle \in \alpha\}$$

defines the supremum of  $\alpha$  in  $\mathcal{S}_0(L)$ . Since  $\overline{\otimes}$  is l.u.b. of the multiplicative parts in  $L^{L \times L}$  it suffices to prove that it is a part of some SMC-structure on  $L$ . The equalities  $x = x \overline{\otimes} 1 = 1 \overline{\otimes} x$ ,  $x \overline{\otimes} y = y \overline{\otimes} x$ , and preservation of suprema are evident. That leaves associativity to be proved.

We have  $(x \overline{\otimes} y) \overline{\otimes} z = \bigotimes_{\alpha \in \alpha} (\bigotimes_{\beta \in \beta} (x \otimes' y)) \otimes z = \bigotimes_{\alpha \in \alpha} \bigotimes_{\beta \in \beta} ((x \otimes' y) \otimes z)$  by (T0). Since  $\alpha$  is a chain this equals  $\bigotimes_{\alpha \in \alpha} \bigotimes_{\beta \in \beta} ((x \otimes y) \otimes z) = \bigotimes_{\alpha \in \alpha} \bigotimes_{\beta \in \beta} (x \otimes y \otimes z)$ . By the same argument (we need (T0)\*, but that is taken care of by the commutativity of  $\otimes$ 's),  $\bigotimes_{\alpha \in \alpha} \bigotimes_{\beta \in \beta} (x \otimes y \otimes z) = x \overline{\otimes} (y \overline{\otimes} z)$ .

2.3. Proposition. If  $L$  is a chain then any  $\varphi: L \rightarrow L$  with properties (i) - (iii) is  $[-, 0]$  for some SMC-structure on  $L$ .

Proof: Put

$$x \otimes y = \begin{cases} 0 & \text{if } x \leq \varphi y \\ x \wedge y & \text{otherwise} \end{cases}$$

and check that it works.

In [5] it is shown that in case of  $L = [0,1]$  and  $\varphi$  an antiisomorphism of  $L$  there is an SMC-structure isomorphic with  $\langle \text{Max}(0, x + y - 1), \text{Min}(1, 1 - y + z) \rangle$  such that  $\Phi = [-, 0]$ .

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