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ON SOME COMBINATORIAL PROBLEMS IN UNIFORM SPACES

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0. We have solved some problem in the theory of uniform spaces using a combinatorial representation of them. The following easy observation is very important for our purpose:

Observation: Let $(X,\mathcal{U})$ be a uniform space. Let $\mathcal{P},\mathcal{Q}$ be $\mathcal{U}$-covers of X such that $\mathcal{P} \overset{*}{\prec} \mathcal{Q}$. Then

$$\bigcup_{x\in Q}\{P\in\mathcal{P} \mid x\in \mathcal{P}\}\supset\{P\in\mathcal{P}\mid P\supset Q\}\supset$$

$$\supset\bigcup_{x\in Q}\{P\in\mathcal{P}\mid \mathrm{st}\,(x,\mathcal{Q})\subset P\} \text{ for each } Q\in\mathcal{Q} .$$

Using Observation we have constructed uniform spaces as complicated as possible, namely we exhibited a class $\mathcal{K}$ of simply described uniform spaces (see [P]) that projectively generate the category UNIF. So these spaces resemble to ℓ_∞'s and really, each member of $\mathcal{K}$ is uniformly homeomorphic to the positive part of the unit sphere of some ℓ_∞ (ℓ_∞ is endowed with the metric uniformity).

Results:

1. Point-character of a uniform space

Definition: $(X,\mathcal{U})$ is a uniform space. A point-character $pc(X,\mathcal{U})$ is the least cardinal m such that there is a base $\mathcal{B}$ of $\mathcal{U}$ such that card $\{P\in\mathcal{P}\mid x\in P\}$ is less than m for each

$x \in X$ and each $\mathcal{P} \in \mathcal{B}$.

Theorem: $\operatorname{pc} \ell_\infty(m) > m$ for each infinite cardinal m. Theorem implies that $\ell_\infty(\omega_0)$ has no point-finite base. Under Generalized Continuum Hypothesis [GCH] , Theorem is the best possible result as $\operatorname{pc} \ell_\infty(m) \leq 2^m$ in general.

2. Cardinal reflections

$(X, \mathcal{U})$ is a uniform space, α is an infinite cardinal. We define $p_\alpha \mathcal{U} = \{ \mathcal{P} \in \mathcal{U} \mid \operatorname{card} \mathcal{P} < \alpha \}$

If [GCH] holds or if $(X, \mathcal{U})$ has a point-finite base, then $p_\alpha \mathcal{U}$ is a uniformity for each α . But applying Baumgartner's theorems on almost-disjoint sets (see [B]) we receive that it is consistent with ZFC to suppose that $p_\alpha \mathcal{U}$ is not a uniformity for any $\alpha \geq \omega_2$ ($p_\alpha \mathcal{U}$ is always a uniformity if $\alpha = \omega_0, \omega_1$)

3. Modification preserving completeness

We are looking for a modification (i.e. a reflection preserving underlying sets) which preserves completeness. We know one : identity: UNIF $\longrightarrow$ UNIF but no other (and maybe, there is no other). The following theorem indicates the complexity of the problem (the problem is due to Z. Frolík).

Theorem: Let r: UNIF $\longrightarrow$ UNIF be a modification. If there is a cardinal m such that $\operatorname{pc}(rK) < m$ for $K \in \mathcal{K}$ ($\mathcal{K}$ is mentioned sub 0) then r does not preserve completeness.

Example: The distal modification does not preserve completeness.

References:

[B] Baumgartner J.E.: Almost-disjoint sets, the dense-set problem and the partition calculus, to appear in Annals of Math. Log.

[P] Pelant J.: Cardinal reflections and point-character of uniformities, Seminar Uniform Spaces 1973-1974 directed by Z. Frolík, MÚ ČSAV.