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ATOMS IN UNIFORMITIES

by

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A list of solved and open problems concerning atoms in lattices of continuous structures is presented.

1. Let U be an ultrafilter on X, let $x_0 \in X$. Denote T_U the topology on X such that $x_0 \in \overline{M} \Longleftrightarrow M \in U$ and such that other points are isolated.

Theorem (folklor). Atoms in the lattice of all topologies on X are just topologies of the form T_U. Each topology is a supremum of atoms.

2. Let U, V be two distinct ultrafilters on X. Denote P_{UV} the proximity on X such that two disjoint sets A, B are proximal iff $A \in U$, $B \in V$ or conversely.

Theorem. Atoms in the lattice of proximities on X are just proximities of the form P_{UV}. Each proximity is a supremum of atoms.

3. Let U be an ultrafilter on X and $f: X \longrightarrow X$ a bijection such that $fU \neq U$. Denote S_U the uniformity a base of which consists of covers of the form $\{\{x, fx\} \mid x \in F\} \cup \cup \{\{x\} \mid x \in X\}$, where $F \in U$.

Theorem [1] . Proximally non-discrete atoms in the lattice of all uniformities on X are just uniformities of the form S_U.

The uniformity S_U induces the proximity $p_{U\ fU}$ and is minimal, but not necessarily the finest one with this property. In other words, S_U need not be proximally fine. Let us consider the following properties of an ultrafilter U on a countable set N.

PF S_U is proximally fine

OPF S_U is proximally fine among all zero dimensional uniformities

Sel U is selective

R for each two maps f, g: $N \longrightarrow N$ such that fU = gU there is $F \in U$ with $f/U = g/U$

P for each two one-to-finite relations f, g: $N \longrightarrow N$ such that fU = gU there is $F \in U$ such that for each $x \in F$ we have $fx \cap gx \neq \emptyset$.

Theorem. Sel $\Longrightarrow$ P $\Longrightarrow$ PF $\Longrightarrow$ OPF $\Longrightarrow$ R

The implication P $\Longrightarrow$ Sel does not hold (A. Louveau, private communication) while the implications PF $\Longrightarrow$ P, OPF $\Longrightarrow$ $\Longrightarrow$ PF are open problems.

4. If U is an ultrafilter on X then denote A_U the uniformity on X consisting of all covers P with $P \cap U \neq \emptyset$.

Theorem [1] A_U is an atom iff U is selective. Each proximally discrete atom refines some A_U.

If U is an ultrafilter on X and a uniformity A_x on $Y \times \{x\}$ is given for each $x \in X$ then all covers of $Y \times X$ of the form $\bigcup \{P_x \mid x \in F\} \cup \{\{x\} \mid x \in Y \times X\}$, where $F \in U$

and P_x is in A_x for each $x \in F$, form a basis of a uniformity which will be denoted by $\sum_U P_x$. If each A_x is an atom so is $\sum_U P_x$. Thus, assuming the existence of selective ultrafilters we dan construct atoms on arbitrary cardinalities.

There exists an example of a proximally discrete atom which is not of the form $\sum_U P_x$.

The following problems remain open: Is every atom zero-dimensional? Given U, how large can be the cardinality of the set of atoms refining A_U ?

References:

[1] Pelant J. and Reiterman J.: Atoms in uniformities, Seminar Uniform Spaces (directed by Z. Frolík), MÚ ČSAV, Praha 1975.