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ON SEMIGROUPS OF σ-ENDOMORPHISMS

by

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Let X be a set with a separable σ-algebra $\mathcal{B}$ of its subsets and with a σ-ideal $\mathcal{J}$ of $\mathcal{B}$. A $\mathcal{B}$-measurable partial transformation $f: D(f) \longrightarrow X$ with $X \supset D(f) \in \mathcal{B}$ is called non-singular if $f^{-1}(A) \in \mathcal{J}$ whenever $A \in \mathcal{J}$.

Every non-singular transformation f induces a σ-endomorphism F of the quotient σ-ring $\mathbf{B} = \mathcal{B}/\mathcal{J}$, the mapping F being defined by $F([A]) = [f^{-1}(A)]$, $A \in \mathcal{B}$. In the other direction it was proved by R. Sikorski (see e.g. Boolean Algebras, Springer Verlag, 1964; Theorem 35.2) that if X is a Borel space (i.e. is Borel isomorphic to a Borel subset of the Hilbert cube) then every σ-endomorphism of $\mathbf{B}$ is pointwise induced by a transformation of X.

Suppose now that, for a Borel X, a semigroup S acts by $s \longrightarrow F_s$ on $\mathbf{B}$ in such a way that $F_{st}(a) = F_s(F_t(a))$ and the F_s are σ-endomorphisms of $\mathbf{B}$. Does there exist some action $s \longrightarrow f_s$ of S on X such that f_s induces F_s for any s ? (As the rule of composition for partial transformations f, g we take $D(fg) = f^{-1}(D(g))$ and $(fg)(x) = g(f(x))$ for $x \in D(fg)$). If the answer is "yes", we say that S is pointwise induced.

In his paper "Point realizations of transformation groups" (Ill. J. Math. 6(1962), 327-335) G.W. Mackey gave the positive answer under the following assumptions:

(a) S is a locally compact second countable group and acts automorphically on $\mathfrak{B}$ by $s \longrightarrow F_s$,

(b) $\mathfrak{J}$ is the ideal of null sets for some finite measure on $\mathfrak{B}$,

(c) for any finite measure m on $\mathfrak{B}$ and for any $A \in \mathfrak{B}$ the real function $s \longrightarrow m(F_s([A]))$ is Borel measurable.

Ours is a different approach as we do not impose any but the algebraic structure on S. Below are listed our results

1. Let S be a countable semigroup acting by $s \longrightarrow F_s$ on $\mathfrak{B}$. If each F_s is pointwise induced (e.g. if X is a Borel space) then S is pointwise induced, too.

2. Let S acting on $\mathfrak{B}$ be a free product of its subsemigroups S_i, $i \in I$. If the S_i are pointwise induced then S is pointwise induced as well.

For the formulation of next results we need a definition:

A $\mathfrak{B}$-measurable nonsingular transformation f is called two-sided nonsingular if $f(A) \in \mathfrak{J}$ for any $A \in \mathfrak{J}$. If, moreover, $D(f) = X$ and f is 1-1 and onto then f is called a point automorphism.

3. Let S be a direct sum of $\aleph_1$ countable semigroups S_α, $\alpha < \omega_1$, all containing the identity element e of S. Suppose S acts by $s \longrightarrow F_s$ on $\mathfrak{B}$ and that

(1) each F_s is induced by a two-sided nonsingular

transformation,

(2) F_e is the identity of $\mathbb{B}$.

Then S is pointwise induced by two-sided nonsingular transformations.

For groups of automorphisms we obtain

4. If a group G acting automorphically on $\mathbb{B}$ is pointwise induced as a semigroup then it is pointwise induced by point automorphisms.

As a corollary of two last results we obtain:

5. Suppose G is a direct sum of $\aleph_1$ countable groups and acts automorphically by $g \longrightarrow F_g$ on $\mathbb{B}$. If all F_g are pointwise induced then G is pointwise induced by point automorphisms.

In particular, using Hamel bases for linear spaces over the field of rationals we get:

6. Let $\aleph_1 = 2^{\aleph_0}$. If X is a Borel space and if $G = R^n$ acts automorphically on $\mathbb{B}$ for some $n = 1,2,\ldots, \aleph_0$ then G is pointwise induced.

We can consider $(\mathbb{B}, \cap)$ a semigroup acting on $\mathbb{B}$ by $b \longrightarrow E_b$ with $E_b(a) = a \cap b$, $a \in \mathbb{B}$.

7. The following conditions are equivalent

(i) the action $b \longrightarrow E_b$ is pointwise induced,

(ii) there is a lower density $D: \mathbb{B} \longrightarrow \mathcal{B}$.

This last result shows, in the light of results of J. von Neumann and M.H. Stone, that if X is an uncountable Borel space and $\mathcal{J}$ the σ-ideal of countable subsets of X then ($\mathbb{B}$, $\cap$) is pointwise induced iff the continuum hypothesis holds.

The proofs are being prepared for publication in Colloquium Mathematicum.