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APPLICATIONS OF VECTOR INTEGRATION TO SPECTRAL THEORY

by

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1. A generalization of Bochner's integral.

The spectral theory requires a theory of integration of functions with values in locally convex spaces which in general are neither metrisable nor separable. The Pettis integral in its full generality is not suitable because the most useful theorems of classical integration theory e.g. Fubini's thm, Lebesgue's differentiation thm, are not valid for general Pettis integrals.

Let T be a topological Hausdorff space, μ a Radon measure on T, and assume T is a countable union of open sets of finite measure.

Definition: A function $f: T \longrightarrow E$, quasi complete l.c.v.s. is totally μ-summable if it is μ-measurable (Lusin property) and if there exists $B \subset E$, bounded closed, absolutely convex, such that $\int |f|_B d\mu < +\infty$.
$(|x|_B \leq \inf\{\lambda, x \in \lambda B\})$

It can be shown that a totally μ-summable function is Pettis integrable and that the integrals $\int_A f d\mu$ belong to E_B the Banach space generated by B. Although $f(t) \in E_B$ a.e. $f: T \longrightarrow E_B$ is not in general Bochner integrable. Nevertheless most of the useful properties of Bochner integral carry over, in particular Fubini's thm and the Lebesgue differentiation theorem e.g. $\frac{d}{dt}\int_a^t f(s)ds = f(t)$ a.e.

To extend the definition to general measure spaces requires the solution of a problem: see Problem book .

2. Given a Hilbert space $\mathcal{H}$ and a selfadjoint or normal operator (T, D_T), with discrete spectrum, there exists a decomposition

$$(1) \qquad \mathcal{H} = \bigoplus_{\lambda \in \sigma(T)} \mathcal{H}_\lambda \qquad \sigma(T) \text{ spectrum of } T$$

with

$$x = \sum_\lambda x_\lambda \quad \text{in } \mathcal{H}$$

$$\|x\|^2 = \sum \|x_\lambda\|^2$$

when the spectrum is not discrete, one can obtain a decomposition $T = \int \lambda E(d\lambda)$ with $E(\ )$ the spectral measure associated with T, but a decomposition closer to (1) is also possible as follows:

Let $\mathcal{E}$ be a quasi-complete locally convex space over $\mathbb{C}$. With every Hilbert subspace $\mathcal{H} \overset{j}{\hookrightarrow} \mathcal{E}$ there is associated a kernel $H = jj^*$ a map from the anti-dual of $\mathcal{E}$, $\mathcal{E}^*$, to $\mathcal{E}$, such that $H(\mathcal{E}^*) \subset \mathcal{H}$ and $(x, He^*) = \langle x, e^* \rangle$. The kernel H is linear Hermitian: $\langle He^*, f^* \rangle = \overline{\langle Hf^*, e^* \rangle}$ and positive $\langle He^*, e^* \rangle \geq 0$. It has been shown by L. Schwartz [2] that conversely there corresponds to every positive kernel $H: \mathcal{E}^* \longrightarrow \mathcal{E}$ a unique Hilbert subspace $\mathcal{H} \subset \mathcal{E}$. The set of positive kernels is a closed convex cone in the real locally convex space of all Hermitian kernels, topologized with the topology of pointwise convergence in $\mathcal{E}^*$; the topology is carried over to the set Hilb($\mathcal{E}$) of Hilbert subspaces of $\mathcal{E}$. If we now put $\mathcal{H}_A = E(A)\mathcal{H}$ the map $A \longrightarrow \mathcal{H}_A$ is a Hilb ($\mathcal{E}$) valued measure which under conditions to be described below has a Radon Nikodym derivative with respect to a basis measure μ .

Theorem. Let $\mathcal{E}$ be a Fréchet space and assume the map $\mathcal{H} \xrightarrow{j} \mathcal{E}$ is absolutely summing. Then given a spectral measure E( ) there exists a measure $\mu \geq 0$ such that $\mu(A) = 0 \Longleftrightarrow \mathcal{H}_A = 0$ and a μ-essentially unique family $\lambda \longrightarrow \mathcal{H}_\lambda \in$ Hilb ($\mathcal{E}$) totally μ-summable such that

$$\mathcal{H}_A = \int_A \mathcal{H}_\lambda \, d\mu(\lambda)$$

This implies that every $x \in \mathcal{H}$ has a decomposition

$$x = \int x(\lambda)\, d\mu(\lambda) \qquad \text{in } \mathcal{E}$$

with

$$\|x\|^2 = \int \|x(\lambda)\|^2_{\mathcal{H}_\lambda} \, d\mu(\lambda).$$

$$E(A)x = \int_A x(\lambda)\, d\mu(\lambda)$$

and

$$Tx = \int \lambda x(\lambda)\, d\mu(\lambda) \qquad x \in D_T$$

if E ( ) is associated with T.

Furthermore if there exists a continuous linear operator $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{E}$ such that $T = \mathcal{T}/_{D_T}$ it follows that

$\mathcal{H}_\lambda \subset \mathrm{Ker}(\lambda \mathcal{I} - \mathcal{T})$ for μ a.e. λ ,

i.e. $\mathcal{T}x = \lambda x \quad \forall x \in \mathcal{H}_\lambda$.

Remark. The condition j absolutely summing cannot be improved in this theorem.

This is closely related to and inspired by the works of

[1] Gel'fand & Shilov: Generalized Functions

[2] L. Schwartz: Sous-espaces Hilbertiens d'espaces vectoriels topologiques et noyaux reproduisants. Journal d'Analyse Mathématique d'Israel 1964.