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# ON ONE GENERALIZATION OF THE WEAKLY COMPACTLY GENERATED B-SPACES

by

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Because reflexive B-spaces are exactly those which are $\sigma$-compact in their weak topology, the following generalizations of $\sigma$-compactness are of interest:

1) analytic topological spaces ( a topological space T is called analytic iff there exists a mapping f from $\omega^\omega$ , $\omega$ being the set of all finite ordinal numbers and also so the first infinite ordinal number into the set of all compacts in T such that for any open G in T the set $\{\pi \in \omega^\omega ; f(\pi) \in G\}$ is open in usual product topology of $\omega^\omega$.

2) Topological spaces which are $K_{\sigma\delta}$ ( a topol. space T is called $K_{\sigma\delta}$ in a topological space $T'$ iff $T \subset T'$ , the topologies of T and $T'$ coincide on T and there are compacts $A_{in}$ in $T'$ so that $T = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_{in}$ ).

It is easy to show that for any topol. space T it holds:

(T is $\sigma$-compact) $\Longrightarrow$ (T is $K_{\sigma\delta}$ in some top. space $T'$) $\Longrightarrow$ (T is analytic) $\Longrightarrow$ (T is Lindelöff in its weak topology).

Definition 1: A B-space X is called weakly analytic (wA) iff X is analytic in its weak topology.

Definition 2. A B-space X is called weakly $\dot{K}$ (WK) iff X is $K_{\sigma\delta}$ in $X^{**}$ (second dual in its $w^*$-topology).

Remark 1. It can be shown that if a B-space X is $K_{\sigma\delta}$

in some uniform space $T'$, then X is WK, so the definition of WK property is not too restrictive.

Proposition 1: Let X be a WCG B-space. Then X is WK (with convex $A_{\lambda\mu}$'s - we will call this CWK property (or space)).

This was proved independently by D. Preiss and Talagrand and by means of this observation they solved this problem of J. Lindenstrauss:

Is every B-space WCG iff it is Lindelöff in its weak topology?

So the implication " $\Longrightarrow$ " holds and the opposite cannot be true because WCG property is not hereditary (on closed linear subspaces) and Lindelöff property is.

Many of basic properties of WCG B-spaces can be proved for CWK B-spaces:

Theorem 1. Let X be a CWK B-space. Then:

(a) X has a projectional resolution of identity i.e. there are linear projections $P_\alpha$, $\omega \leq \alpha = \varkappa$ ($\varkappa$ is the first ordinal number of cardinality dens X = inf {card H ; H is dense in X}) such that:

(i) $\| P_\alpha \| = 1$ for any $\alpha : \omega \leq \alpha = \varkappa$,

$P_\varkappa$ = identity on X ,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for any $\omega \leq \alpha \leq \beta \leq \varkappa$,

(iii) dens $P_\alpha(X) \leq$ card $\alpha$ ,

(iv) the function $P_\alpha(x)$, $x \in X$ fixed, is continuous on $\langle \omega, \varkappa \rangle$ in the usual order topology.

(b) dens X = $w^*$ - dens X (= density in $w^*$-topology)

(c) there exist a set $\Gamma$ and a linear continuous one-to-one mapping from X into $c_0(\Gamma)$.

(d) X admits an equivalent locally uniformly rotund (LUR) norm.

(e) X has an equivalent Fréchet differentiable norm iff X has an equivalent norm which dual norm on $X^*$ is LUR iff X has a shrinking Markuševič basis.

(f) If $X^*$ is also CWK, then X is WCG.

Proofs of this theorem are mainly generalizations of those for WCG B-spaces. They are in some cases more simple. Because CWK property is hereditary, every subspace of WCG B-space is CWK, but while we do not know any definite wey how for given subspace of WCG B-space to find this WCG space, for CWK spaces this ambiguity is overcome by Remark 1.

Problems:

Problem 1. All converses to the following implications:

A B-space X is:

(subspace of WCG B-space) $\Longrightarrow$ (CWK) $\Longrightarrow$ (WK) $\Longrightarrow$ (WA) $\Longrightarrow$ $\Longrightarrow$ (weakly Lindelöff) ?!

Problem 2. Is Theorem 1 true also for X only WK (WA, weakly Lindelöff) ?

Problem 3. Let $X^*$ be a CWK B-space. Is then

a) $X^*$ with Radon-Nikodým property,

b) X Asplund space (i.e. strong differentiability space) ?