Heinrich von Weizsäcker
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ON BARYCENTERS IN NON-COMPACT SETS

by

Heinrich v. WEIZSÄCKER

I. The following theorem is due to G. Winkler (Thesis, München '76). It is an extension of a particular case of the Choquet-Bishop-de Leeuw to the non-compact case. It improves an earlier result of myself (Math. Zeitschrift 1975).

Let T be a completely regular space. Let $C_b(T)$ (resp. $\mathcal{B}(T)$) be the space of all continuous (resp. Borel measurable) real-valued bounded functions on T. Let $\mathcal{M}(T)$ be the space of all bounded Radon measures on T.

Theorem A: If X is a convex $\sigma(\mathcal{M}(T), C_b(T))$-closed bounded subset of $\mathcal{M}_+(T)$, then for each $\mu \in X$ there is a probability measure p on the σ-algebra over ex X generated by the functions $\nu \mapsto \nu(\varphi)$ $(\varphi \in \mathcal{B}(T))$ such that

$$\mu(\varphi) = \int_{\text{ex } X} \nu(\varphi)\, dp(\nu) \qquad \forall \varphi \in \mathcal{B}(T)$$

In particular ex $X \neq \emptyset$.

Problem: Find a proof of ex $X \neq \emptyset$ which does not use Choquet theory.

II. Theorem B: (Fremlin-Pryce, Proc. London. Math. Soc. 1974). Let E be a real locally convex linear space. Let X be a bounded subset of E. Then

X is measure convex $\Longleftrightarrow$	X is a Krein set
(i.e. every Radon measure on X has a barycenter which is in X)	(i.e. if $L \subset X$ is compact, then the closed convex hull of L is compact and contained in X)

Remark: A convex set X is a Krein set, if e.g.: a) X is complete in some (E,E')-topology (Thm. of Krein-Šmulian), b) X is locally compact in the relative topology, c) X is the intersection of Krein sets.

The next theorem shows (by b) and c) in the above Remark) that a convex G_δ set in a compact Z need not be the intersection of convex open subsets of Z. It gives a negative answer to questions of Christensen and Topsøe.

Theorem C: There is a compact convex metrizable subset Z of a locally convex space E, a convex G_δ set X in Z and a probability measure p such that

1 supp $p \subset X$

2 X does not contain the barycenter of p

3 $p(L) = 0$ for all compact convex subsets L of X.

Proof by example: $E = \mathcal{M}([0,1])$ with topology $\sigma(\mathcal{M}([0,1]), C([0,1]))$.

$Z = \{\mu \in E : \mu \geq 0, \mu(1) = 1\}$, λ = Lebesgue measure on $[0,1]$,

$X = \bigcap_{n \in \mathbb{N}} \{\mu \in Z : \lambda + \frac{1}{n}(\lambda - \mu) \notin Z\}$, $p = \varphi(\lambda)$, where $\varphi : [0,1] \ni t \longmapsto \delta_t$.

This example can be embedded into other spaces (e.g. Hilbert space or non locally convex spaces) by

Theorem D Let Z_1 be a compact metrizable subset of a locally convex space E_1 and let Z_2 be a compact convex infinite dimensional subset of a topological linear space E_2. Then there is an affine homeomorphism from the closed convex hull of Z_1 to a subset of Z_2 .

(Thm C + Thm D are contained in a paper of mine submitted to Math. Scand.)