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Submeasure and measure

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SUBMEASURES AND MEASURES

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Let $(X, \mathcal{A})$ be a set with a Boolean algebra $\mathcal{A}$ of subsets. A submeasure φ on $\mathcal{A}$ is a setfunction which has the properties

i) $\varphi(\emptyset)=0$; $A \subseteq B \Longrightarrow \varphi(A) \leq \varphi(B)$

ii) $\varphi(A \cup B) \leq \varphi(A)+\varphi(B)$.

The submeasure φ is pathological if there does not exists a non trivial finitely additive non negative measure dominated by φ . The existence of pathological submeasures has been shown in [1] and independently by Preiss &Vilimovsky and by Popov .

The main open problem in the subject is the control measure problem ,which is equivalent with the following problem .

The submeasure φ is called a Maharam submeasure if it is defined on a σ-field and sequentially point continuous (see [1]) . The control measure problem is equivalent with the problem whether or not every Maharam submeasure admits a probability (countably additive) with the same zero sets. It was shown that if a control measure exists then there exists a control measure dominated by φ . In fact the problem is equivalent with whether or not the Maharam submeasure is pathological.

Our main result so far is that a control measure exist if and only if the Maharam submeasure φ defined on the measurable space $(X, \mathcal{B})$ fulfills the condition:

We consider the unit interval I with usual Borel structure and Lebesgue measure v .Let $A \subseteq X \times I$ be a measurable subset and suppose

$$\forall_{x \in X} : v(\{t \in I \mid (x,t) \in A\})=0 \ .$$

Then there exists a $t_0 \in I$ such that

$$\varphi(\{x \in X \mid (x,t_0) \in A\})=0 \ .$$

If the above statement is true (for every measurable set A) then there is a control measure .The condition is of course necessary (Fubini theorem).The proof that the condition is sufficient also can be found in [2].

It is not known whether a translation invariant Maharam submeasure defined on the Borel subsets of a compact metrizable abelian group is pathological.

It is not known whether a control measure exists for measures taking values in the space of equivalence classes of measurable functions (with topology of convergence in measure).

1) Wojchiech Herer & Jens Peter Reus Christensen,On the existence of pathological submeasures and the construction of exotic topological groups,Math.Ann.213,203-210(1975).

2) Jens Peter Reus Christensen,Some results with relation to the control measure problem,To appear.