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A real Banach space X is an <u>Asplund space</u> provided for every open convex set $U \subset X$, every continuous convex functional $\varphi : U \to \mathbb{R}$ is (Frechet-) differentiable on a dense G_δ subset of U [3]. The main result discussed is the following.

THEOREM (Asplund, Namioka, Phelps, Stegall). <u>The following two statements are equivalent</u>:

(1) X <u>is an Asplund space</u>.

(2) X^* <u>has the Radon-Nikodym property</u> (RNP).

A proof was given by breaking the result into two parts, the first of which is

THEOREM ([3], [4]). Statement (2) above is equivalent to

(3) <u>For every closed bounded set</u> $A \subset X^*$, <u>the set</u>

$$\{x \in X : \tilde{x} \text{ strongly exposes } A\}$$

<u>is a dense</u> G_δ <u>subset of</u> X.

The second part, of course, is to show that (3) and (1) are equivalent; a proof can be found in [3], but here we gave a new proof via the following two easy lemmas.

DUALITY LEMMA. *Let* A *be any closed bounded subset of* $X^* \times \mathbb{R}$, *and let* $\varphi(x) = \sup\{f(x)+\alpha : (f,\alpha) \in A\}$ $(x \in X)$. *Then* φ *is a convex functional on* X *satisfying a Lipschlitz condition. Moreover,* φ *is differentiable at* x *(with gradient* g*) iff the map on* $X^* \times \mathbb{R}$ *sending* (f,α) *into* $f(x)+\alpha$ *strongly exposes* A *(at the point* $(g,\varphi(x)-g(x))$.)

REDUCTION LEMMA. *Let* U *be any open convex subset of* X *and* $\varphi : U \to \mathbb{R}$ *any continuous convex functional. For each* $n = 1,2,\ldots,$ *let*

$$A_n = \{(f,\alpha) \in X^* \times \mathbb{R} : f(x)+\alpha \leq \varphi(x), \text{ all } x \in U, \ \|f\| \leq n, \ |\alpha| \leq n\},$$

and let $\varphi_n(x) = \sup\{f(x)+\alpha : (f,\alpha) \in A_n\}$. *For every* x *in* U *there exist* $\delta > 0$ *and* N *such that*

$$\|y-x\| < \delta \Rightarrow y \in U \quad \text{and} \quad \varphi(y) = \varphi_n(y) \quad \text{for all} \quad n \geq N.$$

References

1. E. Asplund, Frechet differentiability of convex functions, Acta Math. 121 (1968), 31-47

2. R. Huff, preprint.

3. I. Namioka and R.R. Phelps, Banach spaces which are Asplund spaces, Duke Math. J. 42(1975).

4. C. Stegall, in preparation.