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SPACES WITH A BINARY NORMAL SUBBASE

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If (X,d) is a compact metric space then λX denotes the space of all maximal linked systems of closed sets (a sytem of sets is called linked if any two of its members meet) topologized by the metric

$$\bar{d}(\mathcal{M},\mathcal{S}) = \sup_{S\in\mathcal{M}} \min_{T\in\mathcal{S}} d_H(S,T).$$

The space λX is called the superextension of X (cf. DE GROOT [4]). It is known that the superextension of the closed unit segment is homeomorphic to the Hilbert cube Q (cf. van MILL [5]) and it is conjectured that λX is homeomorphic to the Hilbert cube Q iff X is a nondegenerate metrizable continuum (cf. VERBEEK [11]). This conjecture is known as the generalized de Groot conjecture.

If A is a closed subset of X then $A^+ \subset \lambda X$ is defined by

$$A^+ = \{ \mathcal{M} \in \lambda X \mid A \in \mathcal{M} \}.$$

The collection $\{A^+ \mid A \text{ is closed in } X\}$ is a closed subbase for λX. This subbase has two special properties: it is both binary (any of its linked subcollections has nonvoid intersection) and normal (disjoint subbase sets are separated by disjoint complements of subbase sets). A space which has a closed subbase which is both binary and normal is called normally supercompact. Hence each superextension is normally supercompact.

Many results which can be proved for superextensions can also be proved for normally supercompact spaces. This motivates our interest in normally supercompact spaces and in addition, knowledge about normally supercompact spaces is useful if one tries to solve the generalized de Groot conjecture. Many spaces are normally supercompact. Theorem 1 gives a geometric characterization of normally supercompact spaces. Let us first give some definitions. If $x,y,z \in I = [0,1]$ then let $m(x,y,z)$ be the unique point in the set $[x,y]\cap[x,z]\cap[y,z]$. We call a subset X in a product I^κ of unit segment triple-convex provided that for all $x,y,z \in X$ the point p defined by

$$p_\alpha = m(x_\alpha, y_\alpha, z_\alpha) \qquad (\alpha \in \kappa)$$

also belongs to X. We now have the following characterization of normally supercompact spaces.

THEOREM 1(cf. van MILL & WATTEL [9]) <u>A space has a binary normal subbase iff it is compact and can be embedded as a triple-convex set in a product of closed unit segments</u>.

Just as superextensions, normally supercompact spaces have a nice <u>convexity-structure</u>. To show this, let $\mathcal{S}$ be a binary normal subbase for X. A nonempty closed set B in X is called $\mathcal{S}$-<u>closed</u> (or $\mathcal{S}$-<u>convex</u>) provided it is an intersection from members from $\mathcal{S}$. Define

$$H(X,\mathcal{S}) = \{A \subset X \mid A \text{ is } \mathcal{S}\text{-closed}\}.$$

We topologize $H(X,\mathcal{S})$ by regarding it to be a subspace of the hyperspace $H(X)$ of X. Each subset $A \subset X$ is contained in a smallest $\mathcal{S}$-closed set $I_{\mathcal{S}}(A)$, the $\mathcal{S}$-closure of A, i.e.

$$I_{\mathcal{S}}(A) = \bigcap\{S \in \mathcal{S} \mid A \subset S\}.$$

For each point $x \in X$ and $\mathcal{S}$-closed set $B \subset X$ it is easily seen that the set

$$\bigcap_{b \in B} I_{\mathcal{S}}(\{x,b\}) \cap B$$

contains precisely one point, denoted by $p(x,B)$.

THEOREM 2(cf. van MILL & Van de VEL [7]) <u>The mapping</u> $p: X \times H(X,\mathcal{S}) \to X$ <u>is continuous</u>.

This mapping is very useful; it was used in [10] to prove the Lefschetz fixed point property of superextensions, in [6] to prove that a certain subspace of λI is a capset and in [8] to prove the contractibility of some classes of superextensions. It is called the <u>nearest point mapping</u> of X. Notice that if $A \in H(X,\mathcal{S})$ that $p \restriction X \times \{A\}$ is a retraction of X onto A.

Let $\mathcal{S}$ be a binary normal subbase for X and let $\mathcal{T}$ be a binary normal subbase for Y and let $f: X \to Y$ be a continuous surjection. We say that f is <u>convexity preserving</u> (cf. van MILL & WATTEL [9]) provided that $f^{-1}(T) \in H(X,\mathcal{S})$ for all $T \in \mathcal{T}$. It is not hard to see that if $f: X \to Y$ is a continuous surjection that then the induced Jensen mapping (cf. [11]) $\lambda(f): \lambda X \to \lambda Y$ is a convexity preserving mapping with respect to the canonical convexity structures of λX and λY. This implies that if X is a continuum that then $\lambda(f)$ is cellular, even that point inverses of $\lambda(f)$ are AR's. We argue as follows: if X is a (metric) continuum then λX is an AR (cf. van MILL [5]); hence each fiber of $\lambda(f)$ is an AR too being a retract of an AR (cf. theorem 2).

COROLLARY 3: <u>Let</u> $X \approx \varprojlim(X_i, f_i)$ <u>where the</u> f_i's <u>are surjective</u>. <u>Then</u> $\lambda X_i \approx Q$ $(i \in \mathbb{N})$ <u>implies that</u> $\lambda X \approx Q$.

PROOF: Combine results of CHAPMAN [2], [3] and BROWN [1]. □

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