Jerzy Mioduszewski

Compact Hausdorff spaces with two or three types of open subspaces:new results and open questions

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COMPACT HAUSDORFF SPACES WITH TWO OR THREE TYPES OF OPEN SUBSPACES: NEW RESULTS AND OPEN QUESTIONS

Jerzy Mioduszewski (Katowice)

The aim of the lecture is to give a short survey of results and unsolved questions about compact Hausdorff spaces with only two or three topological types among their (non-empty) open subsets. For the brevity, the phrase the space with two, or three, open sets will be used.

Schoenfeld and Gruenhage [7] have shown that if X is a compact (infinite) Hausdorff space with two open sets, then (1) X is dense in itself, and (2) X is totally disconnected. The author showed later [6] that (3) X is perfectly normal, and in consequence, that (4) X has the Souslin property hereditarily. The last means that the discrete subspaces of X must be at most countable.

Easy examples of spaces with two open sets are the Cantor set and the so called double arrow (a closed segment of the lexicographically ordered product $\{0,1\}\times E$, where E is the real line and $0<1$ in $\{0,1\}$).

The square $D\times D$ of the double arrow D is not a space with two open sets, because (4) is not satisfied for $D\times D$, since $D\times D$ contains discrete subspaces of the cardinality that of continuum. Thus, the property to have only two topologically distinct open subsets, is in general not preserved by passing to products.

Eric van Douwen informed the author that there are unorderable compact Hausdorff spaces with two open sets. Namely, the product $D\times C$, where C is the Cantor set, is unorderable (it is even not the continuous image of any compact ordered space, Treybig [8]), is perfectly normal (by a theorem Katětov [5] asserting that the countable product is perfectly normal if all the finite subproducts are; here, $D\times C = D\times \{0,1\}^{\aleph_0}$); there is only one topological type among closed-open subsets of $D\times C$, thus it is easy to check that all non-closed open subsets of $D\times C$ are homeomorphic.

The above examples are separable ones. The existence of non--separable ones contradicts the Martin's axiom and the negation of the Continuum Hypothesis in view of the property (3) (Juhász [4]). However, if there exist homogeneous Souslin lines (they exist if the Axiom of Constructibility is assumed; see the book [2] by Devlin and Johnsbråten, p.39), then the compact non-separable Hausdorff spaces with two open sets can be constructed from these lines as the double arrow was constructed from the real line. Similarly to the earlier comments can be constructed unorderable such spaces.

Although a compact Hausdorff non-separable space with two open sets need not be the Souslin space, there arises the problem:

Problem 1. Implies the existence of compact non-separable Hausdorff spaces with two open sets the existence of Souslin spaces?

The question of the existence of compact Hausdorff spaces with three open sets is much more difficult. I announce here the results due to Witold Bula (a reaserch student of the University in Katowice):

Theorem (Bula). A Hausdorff non-degenerate continuum with (at most) three open sets cannot be metrizable and is perfectly normal.

Thus, by a theorem of Juhász (loco cit.), there do not exist non-separable Hausdorff continuua with three open sets.

Theorem (Bula). Compact metric space with three open sets is dense in itself and totally disconnected, thus is the Cantor set topologically.

This improves the result from [7] . The results by W.Bula will be published separately elsewhere [1] .

Problem 2. Exist there separable non-metric Hausdorff continua with three open sets?

Problem 3. Exist there metric continua with finitely many open sets?

Ryszard Frankiewicz pointed out that there are compact Hausdorff non-connected spaces with three open sets and which are not perfe- ly normal. Thus, the connectedness in the assumptions of the firs theorems cannot be omitted.

Instytut Matematyki
Uniwersytet Śląski
ul. Bankowa 14, 40-007 Katowice, Poland

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