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In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 69--71.

Persistent URL: http://dml.cz/dmlcz/701092

Fifth Winter School of Abstract Analysis, Štěfanová ČSSR 1977,
Organized by the Mathematical Institute of Charles University, Prague.

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We start with a refined version of a convergence principle due to Stephen Simons which admits interesting applications in Choquet's theory concerning arbitrary cones of upper semicontinuous functions on a compact Hausdorff space. Furthermore, in a certain sense our convergence theorem may be incorporated into this context. In view of various consequences it seems remarkable to point out that the theory presented here is not restricted to the Choquet boundary case, it works whenever certain boundary conditions are satiesfied. The following survey contains no proofs. Further information as well as references may be taken from my paper "Varianten zum Konvergenzsatz von Simons und Anwendungen in der Choquettheorie", Arch.Math. 1977.

Let Q denote the set of all sequences $\sigma=(\sigma_n)_n$ with components $\sigma_n \geq o$ such that $\sigma_1+\sigma_2+\sigma_3+\ldots=1$, and let P consist of all those $\sigma \in Q$ which fulfil $\sigma_n=o$ for almost all $n=1,2,\ldots$. In the following X denotes a compact Hausdorff topological space. First we consider a sequence $f_n \in USC(X)$ of upper semicontinuous functions $f_n: X \to [-\infty,\infty[$ which is uniformly bounded above: $f_n \leq c < \infty$ for all $n=1,2,\ldots$. In this situation countable convex combinations are well defined and upper semicontinuous, for each $\sigma \in Q$ define $f^\sigma \in USC(X)$ by $f^\sigma(x) := \sum_{n=1}^{\infty} \sigma_n f_n(x) \in [-\infty, c]$

<u>1. Theorem</u>. Let $Y \subset X$ be a maximum-boundary for $\{f^\sigma : \sigma \in Q\}$. Then $\inf_{\sigma\in P} \max f^\sigma \leq \sup_{y\in Y} \limsup_{n\to\infty} f_n(y)$ and $\limsup_{n\to\infty} \lambda_*(f_n) \leq \sup_{y\in Y} \limsup_{n\to\infty} f_n(y)$ for each Radon probability measure $\lambda \in Prob(X)$.

Next consider a convex cone $T \subset USC(X)$ containing the real constants. For $\lambda,\mu \in Prob(X)$ we write $\lambda \prec \mu$ iff $\lambda_*(f) \leq \mu_*(f)$ for all $f \in T$. In this very general context there are two notions of maximality with respect to $\prec$: $\lambda \in Prob(X)$ is called maximal iff $\lambda \prec \mu$ implies $\mu \prec \lambda$, and strictly maximal iff $\lambda \prec \mu$ even implies $\lambda=\mu$. Of course these notions coincide for example in the case of maximum-stable $T\subset C(X)$ which separates the points of X, in particular in the classical situation of Choquet's theory. Anyway, by Zorn's lemma every $\mu\in\mathrm{Prob}(X)$ is dominated by a maximal $\lambda\in\mathrm{Prob}(X)$. On the other hand, the Hahn-Banach theorem leads to a useful description which works only for strictly maximal measures: For $\lambda\in\mathrm{Prob}(X)$ define $\hat{\lambda}(g):=\inf\{-\lambda_*(f):f\in T, g\leq -f\}$ for all $g\in USC(X)$, then λ turns out to be strictly maximal iff $\lambda_*=\hat{\lambda}$ on USC(X). Further let us introduce the Choquet boundary $Ch(X,T):=\{a\in X:\delta_a\prec\mu \Rightarrow \mathrm{supp}\mu\subset[a]\}$ where $[a]:=\{x\in X:f(x)=f(a)$ for all $f\in T\}$. Finally let us carefully enlarge the given cone:

$$Q(T):=\{f^{\sigma}:\ f_n\in T,\ f_n\leq c<\infty \text{ for all } n=1,2,\ldots\ ,\sigma\in Q\}$$

$$I(T):=\{\inf_n f_n\ :\ f_n\in T \text{ for } n=1,2,\ldots\ ,f_n \text{ pointwise } \downarrow\}$$

Obviously we obtain convex cones such that $\mathbb{R}\subset T\subset Q(T)\subset I(T)$. Now trivial combination of our convergence principle and the definition of $\hat{\lambda}$ yields

<u>2. Lemma</u>. Let $Y\subset X$ be a maximum-boundary for Q(T). And consider $g_n\in USC(X)$ such that $g_n\geq c>-\infty$ for all $n=1,2,\ldots$. Then for each $\lambda\in\mathrm{Prob}(X)$ we have $\inf_{y\in Y}\liminf_{n\to\infty} g_n(y)\leq\liminf_{n\to\infty}\hat{\lambda}(g_n)$.

Applying this lemma to suitable characteristic functions one easily obtains the following generalization of the well known Choquet--Bishop-de Leeuw theorem.

<u>3. Theorem</u>. For a strictly maximal $\lambda\in\mathrm{Prob}(X)$ and an F_σ-subset $Y\subset X$ we have $\lambda(Y)=1$, if Y is a maximum-boundary for Q(T), in particular if $Ch(X,T)\subset Y$.

<u>4. Special Case</u>. Let T separate the points of X and consider an F_σ-subset $Y\subset X$. Then $Ch(X,T)\subset Y$ iff Y is a maximum-boundary for Q(T).

This corollary extends for instance the theorem concerning the existence of the Šilov boundary. As counterexamples even in the classical situation of Choquet's theory show our result cannot be improved very much. However, with some more effort we obtain the following extremely useful characterization of those F_σ-sets which are maximum--boundaries for Q(T).

5. Theorem. For an F_σ-subset $Y \subset X$ the following properties are equivalent:

i) $[a] \cap Y \neq \emptyset$ for all $a \in Ch(X,T)$.

ii) Y is a maximum-boundary for $Q(T)$.

iii) Y is a supremum-boundary for $I(T)$.

iv) For arbitrary $f_n \in T$, $f_n \leq c < \infty$ we have $\inf_{\sigma \in P} \max \tilde{f}^\sigma \leq \sup_{y \in Y} \limsup_{n\to\infty} f_n(y)$.

v) For arbitrary $f_n \in T$, $f_n \leq c < \infty$ and all $\lambda \in Prob(X)$ we have $\limsup_{n\to\infty} \lambda_*(f_n) \leq \sup_{y \in Y} \limsup_{n\to\infty} f_n(y)$.

Since in general situations the existence of suitable strictly maximal measures is rather dubious the following extension of theorem 3 is of interest.

6. Theorem. Consider an F_σ-subset $Y \subset X$ which is a maximum-boundary for $Q(T)$. Then for every $\mu \in Prob(X)$ there is a $\lambda \in Prob(X)$ such that $\mu \prec \lambda$ and $\lambda(Y)=1$.

We conclude with an extended version of the Riesz representation theorem recently found by Benno Fuchssteiner. Our method of proving this theorem as well as the preceding one is based on our results 3 and 5 via a suitable state space embedding. The main idea is not very difficult and may find applications elsewhere.

7. Example. Consider an arbitrary nonvoid set S and a convex cone T of upper bounded functions $f: S \to [-\infty, \infty[$ such that T contains the real constants. The subsequent properties are equivalent:

i) T is a Dini cone: For all pointwise decreasing sequences $(f_n)_n$ in T we have $\inf_{n \in \mathbb{N}} \sup_{s \in S} f_n(s) = \sup_{s \in S} \inf_{n \in \mathbb{N}} f_n(s)$.

ii) Each state on T admits an integral representation: For every additive and positive-homogeneous functional $\lambda: T \to [-\infty, \infty[$ such that $\lambda(f) \leq \sup f$ for all $f \in T$ there exists a probability measure τ on the σ-algebra generated on S by T such that $\lambda(f) \leq \int f d\tau$ for all $f \in T$.