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In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 73–77.

Persistent URL: http://dml.cz/dmlcz/701093

AXIOMATIZABILITY IN FUZZY LOGIC

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Though conceived as a means of dealing with vagueness and uncertainty in the real world, many-valued logic in the first fifty years of its development gradually became a tool utilized primarily for the purposes of two-valued logic. This is perhaps why the introduction of fuzzy sets stimulated such enormous repercussion as we have witnessed in the past decade.x)

The connection between ZADEH's approach to imprecise concepts and the work of ŁUKASIEWICZ and other protagonists of many-valued logic was recognized soon after fuzzy sets made their first appearance in [12] . An interesting attempt at synthesis was made in 1969 by GOGUEN. What makes the complete residuated lattices xx), suggested in [3] as a general basis for the logic of inexact concepts, particularly appealing is that they put under the same roof both the truth-value matrices of ŁUKASIEWICZ ([4]) and complete HEYTING (in particular, BOOLEAN) algebras. The present abstract surveys some results concerning complete-residuated-lattice-valued propositional and predicate calculi which were obtained by the author in 1976/77 .

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x) For a survey of the literature concerning fuzzy sets and systems see [2] .

xx) A residuated lattice consists of a bounded lattice L together wi an isotone binary operation $\otimes$ on L that makes L into a commutative monoid whose unit coincides with the universal upper bound 1 of L , and a binary operation $\rightarrow$ that is isotone in the second and antitone in the first variable. It is assumed that the multiplication $\otimes$ and the residuation $\rightarrow$ are linked by the condition

for any $a , b \in L$, $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

A complete residuated lattice is completely determined by its multiplicative part - the cl_{∞}-monoid $\langle L , \otimes \rangle$. In [3] , yet another term is used - "complete-lattice-ordered semigroup". Here we adhere to the terminology of DILWORTH and WARD ([1]).

Given a complete residuated lattice $\mathbb{L} = \langle L, \otimes, \rightarrow \rangle$ and a formal language $\mathcal{L}$ of zero or first order, the description of formulas of the <u>$\mathbb{L}$-valued calculus in $\mathcal{L}$</u> and the introduction of <u>$\mathbb{L}$-valued models of $\mathcal{L}$</u> may follow the pattern set by classical works in many-valued logic. The many-valued semantics boils down to a set $\mathfrak{S}$ of truth-value functions $T : F \rightarrow L$ where $F = F(\mathcal{L}, \mathbb{L})$ is the set of all formulas. If $\mathcal{L}$ is a zero order language (sentential calculus) these functions arise as unique extensions of valuations $P \rightarrow L$ where P is the set of propositional variables; if $\mathcal{L}$ is a first-order language (predicate calculus) the T's are indexed by $\mathbb{L}$-valued models of $\mathcal{L}$ and valuations of individual variables in their underlying sets. So far, everything is straightforward -- the time for decision comes when one proceeds to define the concepts of satisfaction and validity. The traditional approach that reduces many-valued semantics to a consequence operation $\mathcal{C} : \exp F \rightarrow \exp F$ is all right if the many-valued models in question are to serve the solution of, say, an independence problem in two-valued logic (cf the use of boolean models in SCOTT and SOLOVAY [10]). Nevertheless, if one intends to stick to a particular $\mathbb{L}$ and investigate the $\mathbb{L}$-valued models as legitimate mathematical objects, interesting in themselves, a bivalent notion of validity retains too little information on what actually goes on in the models under consideration to provide adequate means of formal description. Fortunately, there is a natural way of keeping satisfaction and validity fuzzy, and at the same time allowing for <u>fuzzy theories in $\mathcal{L}$</u> represented by L-fuzzy sets $X : F \rightarrow L$ of non-logical axioms. Assigning to every $X : F \rightarrow L$ the fuzzy consequence $\mathcal{C}X : F \rightarrow L$ with the membership degrees defined by the formula

$$(\mathcal{C}X)\varphi = \bigwedge \{ T\varphi \mid T \in \mathfrak{S}, \ T\psi \geq X\psi \text{ for each } \psi \in F \}$$

we obtain a unary operation $\mathcal{C}$ on the complete lattice L^F

which is inflationary, isotone and idempotent, and therefore a likely candidate for a generalization of the two-valued consequence operation, as introduced by TARSKI ([11]), to the complete-lattice-valued case.

Naturally, if one wants to axiomatize the given calculus in such setting one has to devise a fuzzy set of logical axioms and fuzzy rules of inference which would match the many-valued semantics as well as the syntax of the classical propositional (resp. predicate) calculus matches its two-valued semantics. An attempt in this direction was made in [5] where we defined abstract complete-lattice-valued syntax and discussed its basic properties. For instance, the fuzzy modus ponens for the $\mathbb{L}$-valued calculus has the form

$$\frac{\varphi, \varphi \Rightarrow \psi}{\psi} \left(\frac{a\ ,\ b}{a \otimes b} \right)$$

("if φ has been established in degree a and $\varphi \Rightarrow \psi$ in degree b conclude that ψ is valid at least in the degree $a \otimes b$ ").

Now the general problem of axiomatizability reads as follows :

Given $\mathbb{L}$ and $\mathcal{L}$ does there exist a fuzzy set $A : F(\mathcal{L},\mathbb{L}) \to L$ of logical axioms and a set $\mathcal{R}$ of L-valued rules of inference so that for any fuzzy set $X : F \to L$ and any formula $\varphi \in F$ the degree $(\mathcal{C}X)\varphi$, in which φ logically follows from X , equals exactly the join in L of all the degrees of validity derivable for φ from the values of A and X on F by means of the rules included in $\mathcal{R}$?

So far, the above question has been answered (for propositional calculus in [7] and for predicate calculus in [8]) in the case when the underlying lattice is either a finite chain, where the answer is always affirmative x), or the unit interval of reals, where it turns out that an $\mathbb{L}$-valued calculus in $\mathcal{L}$ is axioma-

x) the special case $L = \{0,1\}$ yields GÖDEL's completeness theorem (the two-element boolean algebra is the only residuated lattice on $\{0$

tizable iff $\mathbb{L}$ is isomorphic to the Łukasiewicz residuated interv $\langle I, \otimes, \rightarrow \rangle$ where

$$a \otimes b = 0 \vee (a+b-1) \quad , \quad a \rightarrow b = 1 \wedge (1-a+b) \; .$$

Moreover, the result concerning the interval-valued logic of Łukasiewicz remains valid if we enrich the truth-value algebra by countably many Lipschitz operations on I .

The technique employed in our proofs of the completeness theor leans on two ideas of RASIOWA and SIKORSKI ([9]) : 1. the upper estimates of the semantically induced degrees $(\mathcal{C}X)\varphi$ are obtained by an ultrafilter trick which reduces a LINDENBAUM-TARSKI algebra to the algebra of truth values ; 2. when dealing with predicate calculi, a topological argument on an ultrafilter space ensures that the reduction is well-behaved with respect to quantifiers.

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