Gerd Rodé

A Dini principle for convex functions and the theorem of James

In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 91–93.

Persistent URL: http://dml.cz/dmlcz/701099

Fifth Winter School of Abstract Analysis, Stefanova CSSR 1977,
Organized by the Mathematical Institute of Charles University, Prague.

A Dini Principle for Convex Functions and the Theorem of James

Gerd Rodé

We show that a set T in the unit ball of a dual Banach space E' gives us a good information about the whole space if

Definition: T supports the norm on E:

To each $x \in E$ there exists $f \in T$ such that $f(x)=\|x\|$.

Example: T= extreme points of the unit Ball.

(But note that $T \cap$ Ball $E' = \emptyset$ is possible.)

The following facts concerning such subsets have already been known:

(1) (James) If the unit ball of a Banach space F, considered as a subset of F'', supports the norm on F', then F is reflexive.

(2) (Simons) If $(x_1, x_2 \ldots)$ is a bounded sequence in E such that $f(x_n) \to o$ for each $f \in T$, then $x_n \to o$ weakly.

This was proved by Rainwater in the case T= ex Ball E', using Choquet theory.

We prove (1), (2) and further results using a lemma about sublinear functionals on l_1^+ which generalizes the essential idea in James' proof of (1):

(3) Lemma. Let β be a sublinear functional on l_1^+, $c>o$.

Then there exists a sequence $(q_1, q_2 \ldots)$ of points in l_1^+, $q_n = (q_n^1, q_n^2 \ldots)$, such that

(i) $\|q_n\| = 1$, $q_n^k = o$ if $1 \leqq k < n$.

(ii) If $p: l_1^+ \to \mathbb{R}$ is sublinear, $p \leqq \beta$ and $p(q_1) = \beta(q_1)$, then $p(q_n) \geqq \beta(q_1) - c$ for each n.

The proof of (3) is elementary but difficult.

The following theorem does not contain the full power of (3), but it is sufficient for many applications ((4)→ (2),(5),(7)). And it is very easy ~~to work with~~ it.

(4) (Dini principle for convex functions)

Let A be a σ-convex subset of a TVS, $(v_1, v_2 \ldots)$ a sequence of bounded convex functions such that

(i) $v_1 \leqq v_2 \leqq v_3 \leqq \ldots$

(ii) If $a \in A$, then there exists $n \in \mathbb{N}$ with $v_n(a) = v_\infty(a) \left(= : \sup_{l \in \mathbb{N}} v_l(a)\right)$.

Then $\inf v_n(A) \to \inf v_\infty(A)$.

(5) If T is strongly separable, then E' is strongly separable, and the convex hull of T is strongly dense in Ball E'.

(5) contains (1) if F is separable. Another application: The Banach space of all trace class operators on a Hilbert space has the RNP. (You have to show that each separable space of compact operators on the Hilbert space has a separable dual.)

(6) If A is a convex subset of E and if for each sequence $(x_1, x_2 \ldots)$ in A there exists $x_\infty \in A$ with $\liminf_{n\to\infty} f(x_n) \leqq f(x_\infty) \leqq \limsup_{n\to\infty} f(x_n)$, $f \in T$, then A is weakly compact.

(6) is stronger than (1).

(7) If T is the countable union of weak-*-compact sets, then to each $g \in E'$ there exists a nonnegative regular Borel measure on representing g.

Let us finally note that it is possible to deduce characterization of strong compactness with the same methods. For example:

(8) A closed set K in a Banach space is strongly compact iff each continuous seminorm attains its supremum on A.