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Uniform spaces with easy behavior with respect to coreflections

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All uniform spaces are assumed to be separated, R stands for a r line, H(A) for a hedgehog over a set A (that is a cone over a uniformly discrete space A). All coreflections are assumed to be non-trivial, thus all coreflections contain all uniformly discrete spaces. We shall denote by **d** the coreflector onto uniformly discrete spaces, δ the coreflector onto proximally discrete spaces, t_f onto topologically fine spaces, **a** onto Alexandrov spaces. The last coreflection assigns to each uniform space X the coarsest uniformity finer than X and containing all finite cozero covers (see [F]). For any space X and coreflector F we shall denote X-F the class of all spaces Y such that any uniformly continuous $f:Y \longrightarrow X$ remains uniformly continuous into FX. It is well known (see [V]) that X-F is a coreflective class.

The aim of this note is to present a construction of spaces having the property that each coreflector behaves on them either identically or as **d**. The obtained results have some interesting consequences we want to mention shortly about. The details and proofs will appear elsewhere.

<u>Definition</u>: Let $\{k_n\}$ be a sequence of natural numbers. We define a space $D(\{k_n\})$ on a set

$$\{\langle n,i\rangle ;n\in N,\ 1\leq i\leq k_n\}$$

taking $\{\mathcal{U}_m;\ n\in N\}$, where

$$\mathcal{U}_m = \{\{\langle k,i\rangle\}\ ;\ k\leq n\}\cup\{\{\langle k,i\rangle\ ;\ i\leq k_k\}\ ;\ k>n\}$$

for a basis of uniformity.

Setting $k_n = m$, we denote the corresponding space D_N and for $k_n = 2$ we denote the space D_2.

One can easily see that spaces $D(\{k_n\})$ are complete, metrisable, zerodimensional and topologically discrete.

Proposition 1: Let $\mathcal{E}$ be a coreflective subcategory of uniform spaces, F the corresponding coreflector, $\{k_n\}$ a sequence of natural numbers. Suppose $D(\{k_n\}) \notin \mathcal{E}$, then $FD(\{k_n\})$ has a discrete proximity. (All pairs of disjoint sets are proximally far).

That means that taking any sequence $\{k_n\}$, then either $FD(\{k_n\}) = D(\{k_n\})$ or $FD(\{k_n\})$ is finer than $\delta D(\{k_n\}) = aD(\ k_n\ )$. Moreover if $\{k_n\}$ is bounded, then either $FD(\{k_n\}) = D(\{k_n\})$ or $FD(\{k_n\})$ is uniformly discrete. We obtain the following

Corollary 1: The following properties of a uniform space X are equivalent:

(1) X is $D(\{k_n\}) - \delta$ for all sequences $\{k_n\}$.

(2) X is $D(\{k_n\}) - d$ for all bounded sequences $\{k_n\}$.

(3) X is $D_N - \delta$

(4) X is $D_2 - d$

Having any coreflective class $\mathcal{E}$ in uniform spaces, the obtained result gives that either $\mathcal{E}$ is contained in D_2-d, or $\mathcal{E}$ containes the coreflective hull coref(D_2) of $\{D_2\}$. One may find interesting that coref(D_2) is very "large", in fact it containes all metrisable spaces, what follows from the following easy statement (cf. [Č]):

For M,S metrisable, $f: M \longrightarrow S$ is uniformly continuous if and only if fg is uniformly continuous for all $g: D_2 \longrightarrow M$ uniformly continuous.

One may go a step further from the Proposition 1 proving:

Proposition 2: Take any coreflector F in uniform spaces, then either $FD_N = D_N$ or $FD_N = \delta D_N$ or $FD_N = dD_N$.

Instead of D_N we can take any space $D(\{k_n\})$ for an unbounded sequence $\{k_n\}$ of natural numbers. Again similar conclusions as for D_2

Corollary 2: Let $\mathcal{E}$ be any coreflective class in uniform spaces and let neither D_N nor δD_N be in $\mathcal{E}$. Then $\mathcal{E}$ is a subclass of D_N-d.

More interesting results can be obtained if we restrict ourselves to coreflective classes closed under subspaces. We recall that for any coreflective class $\mathcal{E}$ the class Sub($\mathcal{E}$) of all subspaces of spaces in $\mathcal{E}$ forms again a coreflective class (see [V]). A similar theorem for the class Her($\mathcal{E}$) of spaces being hereditarily in $\mathcal{E}$ is not valid in general, but fortunately in the case of D_2-d we obtain again a coreflection having even a nice description:

Theorem 1: The following properties of a uniform space X are equivalent:

(1) X is hereditarily D_2-d

(2) X is hereditarily $D(\{k_n\})$ - d for all bounded $\{k_n\}$.

(3) X is hereditarily $D(\{k_n\})$ -δ for all $\{k_n\}$.

(4) Xis hereditarily D_N - δ

(5) Each countable uniformly discrete union of boundedly finite uniformly discrete families is uniformly discrete.

(6) X is $H(\omega)$-a

(7) X is hereditarily R-a

(8) For any countable family $\{f_n\}$ of uniformly bounded and uniformly continuous real valued functions on X with $\{\text{supp } f_n\}$ uniformly discrete, the function $\sum' f_n$ is uniformly continuous.

(9) For every $Y \subset X$, $f: Y \longrightarrow R$ uniformly continuous, $g: R \longrightarrow R$ continuous bounded, the function gf is uniformly continuous.

Spaces being hereditarily D_N-d have again very nice properties and form a coreflective class. These spaces are studied in [FPV] . We recall at least some most interesting properties of them:

Theorem 2: The following properties of a uniform space X are equivalent

(1) X is hereditarily D_N-d

(2) X is $H(\omega) - t_f$

(3) For every $Y \subset X$, the set $U(Y)$ of all uniformly continuous real valued functions is a ring.

(4) For any sequence $f_n \in U(X)$ such that f_n are bounded and the family $\{supp\ f_n\}$ is uniformly discrete, the sum $\sum f_n$ is uniformly continuous.

(5) $U(X)$ is a ring and for any $Y \subset X$, $f \in U(Y)$, there exists an extension $\bar{f} \in U(X)$ of f.

We shall denote these two coreflections $H(\omega) - a$ and $H(\omega) -$ respectively.

In order to make some conclusions from the remark after Proposition 1, we must know, what is $Sub(coref(D_2))$. It is clear that it is a very large coreflective class containing all metric spaces. Under some set theoretic assumptions, $Sub(coref(D_2))$ may be even the class of all uniform spaces. Assuming [SEQ], the nonexistence of Mazur's sequential cardinals, then $coref(D_2)$ is productive (see [H]), hence $Sub(coref(D_2))$ contains all uniform spaces. Thus under this assumption we have:

<u>Theorem 3</u>: [SEQ] The class $H(\omega) - a$ is the largest nontrivial hereditary coreflective subcategory of uniform spaces.

Further application of our construction may be the following, suggested by Corollary 2: Having any class $\mathcal{A}$ of uniform spaces closed under subspaces. If neither D_2 nor δD_N are in $\mathcal{A}$, then whenever $H(\omega)-t_f \subset \mathcal{A}$, then $H(\omega)-t_f$ is the largest coreflective class contained in $\mathcal{A}$. For example we can prove the following:

<u>Theorem 4</u>: $H(\omega)-t_f$ is the largest coreflective class contained in the following classes:

(a) The class of all X such that for any subspace Y of X, $f \in U(Y)$, there is an extension $\bar{f} \in U(X)$ of f.

(b) The class of all X such that for every free uniform measure μ on X the support supp($\check{\mu}$) of the corresponding Radon measure on the Samuel compactification $\check{X}$ of X lies in the completion $\hat{X}$ of X.

(c) The class of all spaces with the property that each bounded subset of it is precompact.

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