WSAA 5

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In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 119–[120a].

Persistent URL: http://dml.cz/dmlcz/701105

Fifth winter school

A note on the nonexistence of the Feynman integral

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There is the following well known difficulty in the theory of Feynman integral: no measure can be related on $R^{\langle 0,1\rangle}$ to the formal expression $e^{i\int_0^1 [x'(t)]^2\, dt}\, \mathcal{D}(x)$, contrary to the case when the formal expression $e^{-\int_0^1 [x'(t)]^2\, dt}\, \mathcal{D}(x)$ leads to the well defined Wiener measure.

This was first pointed out by Gelfand and Cameron.

The same difficulty arises in the case of the operator valued Feynman integral, introduced by Cameron and Storvick.

Definition. A dynamical system $T = \{ T_s^t ,\ 0 \le s \le t \le 1 \}$ on $L^p (X,\mu)$ is a family of bounded operators on L^p satisfying

(1) $T_t^u \circ T_s^t = T_s^u$, T_t^t = identity

(2) $T_t^{(\cdot)}$, $T_{(\cdot)}^t$ are Borel measurable operatorvalued functions.

For any dynamical system T we can construct the "dynamical operatorvalued Feynman integral" $\vec{\mu}_T$, defined for each $0 = t_0 \le t_1 \le \dots \le t_n = 1$ and each rectangle

$A = A_0 \times A_1 \times \dots \times A_n \subset X^{\{t_0, t_1, \dots t_n\}}$ $X^{\langle 0,1\rangle}$

with A_i Borel by $\vec{\mu}_T(A) = I_{A_n} \circ T_{t_{n-1}}^{t_n} \circ \dots T_{t_0}^{t_1} \circ I_{A_0}$

where I_{A_i} denotes the operator of multiplying by χ_{A_i} .

Examples: 1) The "Feynman" dynamical system defined by the semigroup of operators on $L^2(R)$ related to the Schrödinger equation $\frac{\partial}{\partial t} \varphi = i \Delta \varphi - U \cdot \varphi$

2) The "Wiener" dynamical system, related to the equation $\frac{\partial}{\partial t}\varphi = \Delta\varphi$

It is the aim of this note to investigate the question, when $\vec{\mu}_T$ extents to a vector measure (with values in $L(L^p, L^p)$ - the space of bounded operators on L^p).

The results are the following:

<u>Definition 1</u>. Let $T \in (L^p, L^p)$. Consider L^p with its natural norm and lattice structure.

If $\varphi \geq 0$, put
$$|T|\varphi = \sup_{\substack{\sum\varphi_n \leq \varphi \\ 0 \leq \varphi_n}} \sum |T\varphi_n|$$
iff it exists in L^p. For anarbitrary $\varphi \in L^p$, put
$$|T|\varphi = |T|\varphi^+ - |T|\varphi^- \quad \text{whenever it is defined.}$$
Clearly T is a linear operator (the absolute value of T).

As it will be shown, it often happens that $\mathcal{D}(|T| = \{0\})$.

<u>Theorem 1</u>. Consider the space $L(L^p, L^p)$ with its strong operator topology. If $T \in L(L^p, L^p)$, define $\vec{\mu}_T$ (the "operator integral" on $X \times X$) on Borel rectangles by
$$\vec{\mu}_T(A \times B) = I_B \circ T \circ I_A .$$
Then $\vec{\mu}_T$ can be extended to a vector measure on Borel subsets of $X \times X$ iff $|T|$ is a bounded operator.

Moreover, then there is a Borel measurable function G with $|G| = 1$ such that
$$\vec{\mu}_T = G\,\vec{\mu}_{|T|} .$$
Now we give the extension of Theorem 1 for dynamical systems:

<u>Definition 2</u>. Let T be a dynamical system.

Let each $|T_s^t|$ be bounded and let for each $s < t$
$$\left\{|T_{t_n}^t| \circ \cdots \circ |T_s^{t_1}|,\ s \leq t_1 \leq \ldots \leq t_n \leq t\right\} \text{ be bounded in } L(L^p, L^p).$$

We can define

$$|T|_s^t = \sup |T_{t_n}^t| \circ \dots \circ |T_s^{t_1}| \varphi \quad \text{for each } \varphi \geq 0.$$

It can be cheched that $|T| \overset{def}{=} \{|T|_s^t\}$ (the absolute value of T) is a dynamical system.

Consider also the "truncated dynamical systems" ${}_s^tT$ defined by: ${}_s^tT_{s'}^{t'} = T_{s'}^{t'}$ whenever $s', t' \leq s$ and $s', t' \geq t$,

$${}_s^tT_{s'}^{t'} = Id \quad \text{whenever } s \leq s,\ t' \geq t.$$

Now, the main result says:

<u>Theorem 2</u>. All dynamical integrals $\overrightarrow{\mu_{{}_s^tT}}$ extend to a vector measure on Borel subsets of $X^{\langle 0,1\rangle}$ iff $|T|$ exists. Moreover, then there is a Borel measurable function G on $X^{\langle 0,1\rangle}$ with $|G| = 1$ such that $\overrightarrow{\mu_T} = G.\overrightarrow{\mu_{|T|}}$.

<u>Examples</u>. Let T be an operator on L^p (m) (m-Lebesgue) measure invariant with respect to shipts.

Then $|T|$ exists iff T can be expressed by a convolution with a finite measure.

If $T = \{T_s^t\}$ is a dynamical system defined by a semigroup, invariant with respect to shifts, then if $|T|$ exists, then each $|T|_o^t$ can be expressed by a convolution with $e^{\alpha t}\mu_t$ where μ_t is an infinite divisible probability.

Thus we see the striking difference between the Wiener and Feynman dynamical system.