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Some combinatorial questions related to measure theory

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1. In this lecture we consider a set X, an algebra $\mathcal{A}$ of subsets of X, a set function $\varphi : \mathcal{B} \longrightarrow [0,\infty[$ defined on a subfamily of $\mathcal{A}$ and (finitely additive) measures on $\mathcal{A}$, which dominate φ or are dominated by φ. We define

$$\alpha(\varphi) = \sup \{\mu X \mid \mu \text{ measure on } \mathcal{A},\ \mu \le \varphi \text{ on } \mathcal{B}\}$$
$$\beta(\varphi) = \inf \{\nu X \mid \nu \text{ measure on } \mathcal{A},\ \varphi \le \nu \text{ on } \mathcal{B}\}$$

2. Let k be an integer. A finite sequence $\mathcal{C} = (A_1,\dots,A_m)$ of elements of $\mathcal{B}$ is called a k-fold covering (exact covering, matching, respectively) of X by elements of $\mathcal{B}$, if the sum $\sum_{i=1}^{m} 1_{A_i}$ of the characteristic functions is greater than (equal to, smaller than) $k \cdot 1_X$ (cf.[2],p.419) We define $s(\mathcal{C},\varphi) = \frac{1}{k}\sum_{i=1}^{m} \varphi A_i$.

<u>Theorem 1</u> There are measures μ^* and ν^* with

a) $\alpha(\varphi) = \mu^* X = \inf \{s(\mathcal{C},\varphi) \mid \mathcal{C} \text{ is a multiple covering of } X\}$

b) $\beta(\varphi) = \nu^* X = \sup \{s(\mathcal{C},\varphi) \mid \mathcal{C} \text{ is a multiple matching of } X\}$

c) If φ is monotone and $\mathcal{B} = \mathcal{A}$ we need only consider exact coverings in a) and b).

This fact is contained in a more special form in [4],[6],[7]. Let us emphasize that it provides a connection between measure theory and combinatorics ($\mathcal{C}$ is just a hypergraph).

3. Theorem 1 yields a weak generalization of the well-known maximal network flow theorem of Ford and Fulkerson. Given a network $G=(V,E)$ with source q, sink s and capacity function $c : E \longrightarrow [0,\infty[$ let X denote the set of all elementary paths [2] from q to s. To an edge e corresponds the set $B_e = \{w \in X \mid e \text{ is in } w\}$. Let $\mathcal{B} = \{B_e \mid e \in E\}$ and $\varphi B_e = c(e)$. Flows in the graph-theoretic sense correspond to measures on $\mathcal{P}(X)$ dominated by φ on $\mathcal{B}$. By theorem 1, the maximal flow in G is the infimum of capacities of "multiple cuts" (k-fold cut = sequence of edges which meets every path in X at least k times). This is true for multicommodity networks and infinite networks, too.

Ford-Fulkerson's theorem on "simple cuts" follows from the special structure of usual networks: every k-fold cut splits into k disjoint simple cuts.

4. For every φ, $\alpha(\varphi)=\alpha(\eta)$ and $\beta(\varphi)=\beta(\psi)$, where η and ψ are the outer and inner measure on $\mathcal{A}$ generated by φ. Thus, in the following η denotes a submeasure and ψ a superadditive set function which are normalized: $\eta X=\psi X=1$. On infinite algebras $\mathcal{A}$ there exist ψ with $\beta(\psi)=\infty$, and non-trivial examples of η with $\alpha(\eta)=0$ (so-called pathological submeasures) were given by Popov [6] , Herer and Christensen [3] and Topsøe [7] . For $X=X_n=\{1,\dots,n\}$ and $\mathcal{A}=\mathcal{P}(X_n)$, however, $\alpha(\eta)\geq\frac{1}{n}$ and $\beta(\psi)\leq n$ clearly holds. Hence, the following numbers seem to be of interest.

α_n = inf $\{\alpha(\eta) \mid \eta$ normalized submeasure on $\mathcal{P}(X_n)\}$

β_n = sup $\{\beta(\psi) \mid \psi$ normalized and superadditive on $\mathcal{P}(X_n)\}$

At the 3rd Winter School in Stefanova, 1975, Vašak and Preiss discussed the numbers α_n and raised the question: Which is the first number n with $\alpha_n\neq\frac{n}{2(n-1)}$? We think it is eleven but can only prove it lies between 6 and 11 . Asymptotic behavior of α_n is easier determined.

<u>Theorem 2</u> a) $1\leq\underline{\lim}\ \alpha_n\cdot\log n\leq\overline{\lim}\ \alpha_n\cdot\log n\leq 2\cdot\log 2$

b) $\lim\ \beta_n:\sqrt{n}=1$

5. The proof of theorem 2a in [1] uses the fact that for a submeasure η on $\mathcal{P}(X_n)$ with small $\alpha(\eta)$ there exist large sets with small η-values and small sets with large η-values. This fact also implies for an arbitrary η:

<u>Theorem 3</u>

$$\beta(\eta)\geq\alpha(\eta)\cdot\exp\left(\frac{1}{\alpha(\eta)}-2\right)\quad\text{for } \alpha(\eta)\neq 0$$
$$\beta(\eta)=\infty\quad\text{for } \alpha(\eta)=0$$

The last assertion may be considered as a contribution to the well-known question of Maharam, wether for every continuous submeasure on a σ-algebra of sets there is an σ-additive measure with the same zero-sets. By theorem 2 of [3] and theorem 4 of [6], this question is equivalent to the problem, wether all pathological submeasures are discontinous. This concerns sequential continuity with respect to order-convergence, but it suffices to show that pathological submeasures are not exhaustive, that means, there is a disjoint sequence $(A_i)_{i=1,2,..}$ of elements of with $\eta A_i \geq \varepsilon$ for all i and a certain positive number ε. The above assertion is much weaker, of course. It only implies the existence of a disjoint sequence (A_i) with $\sum \eta A_i = \infty$.

6. Let us present a combinatorial question. A positive answer to that question would imply the statement, that for every pathological submeasure η on $\mathcal{A}$ and every integer n there are n pairwise disjoint sets in $\mathcal{A}$ with η-value $\geq \frac{1}{3}$. (This is near to a positive solution of Maharam's question.)

Let $M_1, M_2, \ldots, M_n$ be subsets of a set X. We assume that the intersection of (d+1) different M_i-s is always empty. Let γ be a subfamily of $\mathcal{P}(X)$ with the following property: if $A \subseteq M_i$ $(1 \leq i \leq n)$ then $A \in \gamma$ or $M_i - A \in \gamma$. Let $q(\gamma)$ be the maximal cardinality of a disjoint family of elements of γ.

Given n and d determine $q = \min \{ q(\gamma) \mid X, M_i \text{ and } \gamma \text{ as above} \}$

q is not greater than $\frac{n}{d}$. Is it equal to $\frac{n}{d}$? (This is true for d=2). Does there exist a positive number δ with $q \geq \delta \cdot \frac{n}{d}$?

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