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A short survey on stable convex sets

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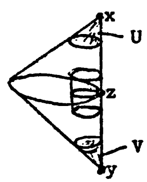
The aim of this talk is to give a short survey on a class of convex sets in l.c. spaces, which has come to interest in the last few years.

Definition: Let K be a convex subset of a l.c.s.. K is called stable, if the midpoint-map $m : K\times K \to K$ is open.

For the whole space, this is of course always true, but not so for all convex subsets. Here is an example:



$\frac{U+V}{2}$ is not a neighborhood of $\frac{x+y}{2} = z$.

Proposition: The following are equivalent:

(1) K is stable.

(2) The map $K\times K\times[0,1] \to K$, $(x,y,\lambda) \to \lambda x+(1-\lambda)y$, is open.

(3) For any convex subset C of K, the (relative) interior of C is convex.

(4) For any open subset U of K, the convex hull of U is open.

Remark: K stable $\Rightarrow$ ex K closed.

Follows from ex $K = K\setminus m(K\times K\setminus\Delta_{K\times K})$. The double-cone above has a non-closed extreme point set.

From now on, assume K to be compact.

<u>Theorem:</u> (Vesterstrøm, O'Brien, Eifler, Uhlenbrok, Debs; 1976):
The following are equivalent:

(1) K is stable.

(2) The resulting map $r : M^1_+(K) \to K$ is open.

(3) $r_{|Max(K)}$ is open (Max(K) := maximal measures).

(4) For all $f \in C(K) : \hat{f} \in C(K)$.

The theorem implies, that Bauer-simplexes $M^1_+(X)$, (X compact) are stable In fact, this is the essential step in the proof. It is also true that for every Hausdorff space X, the set $M^1_+(X)$ of all Radon probability measures in the topology $\sigma(M^1_+(X), C_b(X))$ is stable.

The above conditions are often hard to check. One is in a better situation in the

Finite dimensional case.

Let $K^{(n)} = \{x \in K : \dim \text{face } (x) \leq n\}$ denote the <u>n-skeleton</u>. - E.g. $K^{(o)} = \text{ex } K$. It is easy to show, that K stable $\Rightarrow K^{(n)}$ closed for all $n = 0,1,2,\ldots$.

If $K \subset \mathbb{R}^n$, then $K^{(n)} = K$, $K^{(n-1)} = \partial K$, and $K^{(n-2)}$ are always closed.

<u>Theorem:</u> (Papadopoulou; 1977):
For $K \subset \mathbb{R}^n$ are equivalent:

(1) K is stable.

(2) The correspondence $x \mapsto \text{face } (x)$ is l.c.s.

(3) All skeletons of K are closed.

Reiter-Stavrakas pr ved the eq ivalence:

(4) The space $\mathcal{F}_K$ of faces of K in the Hausdorff-metric is compact.

Corollary:

(1) All $K \subset \mathbb{R}^2$ are stable.

(2) If $K \subset \mathbb{R}^3$: K stable <=> ex K closed.

(3) If $K \subset \mathbb{R}^n$ is a polytope or stricty convex, then K is stable.

(4) (Jamison): The set $\operatorname{conv}\{\Gamma(t) : t \in [0,1]\}$ is stable, where $\Gamma(t) = (t, t^2, \ldots, t^n)$ is the moment curve.

Stability is preserved under finite direct sums, products, open affine maps, affine retractions.
This gives also infinite-dimensional examples, but no good characterization is known for them.

Applications:

The significance of stable convex sets rests on the fact, that a surjection $f : X \to Y$ is open if and only if the correspondence $f^{-1} : Y \to X$, $y \mapsto f^{-1}(y)$, is l.c.s. This allows to apply selection theorems of Michael, Lazar, and Lazar-Lindenstrauss.
The metrizability hypotheses in the following come from these selection theorems.

(A) Extremal operators

Let S be a Choquet simplex.
$\mathcal{A}(S,K)$ = set of all affine, continuous maps $S \to K$.

Theorem: (Papadopoulou and Clausing):
If K is stable and metrizable, then
$\operatorname{ex} \mathcal{A}(S,K) = \{T \in \mathcal{A}(S,K) : T(\operatorname{ex} S) \subset \operatorname{ex} K\}$.

Counterex.: There is a simplex S, such that for all compact, metrizable, infinite X there is
$T \in \operatorname{ex} \mathcal{A}(S, M_1(X))$ with $T(\operatorname{ex} S) \not\subset \operatorname{ex} M_1(X)$.

Remarks:

(1) Using operator representation theorems one obtains from the above theorem a characterization of the extreme operators from certain Banach spaces into simplex spaces as "nice" operators.

(2) Theorem (Cl.): The space $\mathcal{A}(S,K)$ in the uniform topology is itself stable, if K is stable and metrizable.

(B) A Dirichlet problem

A(K) := affine maps in C(K).
A closed subspace H with $A(K) \subset H \subset C(K)$ is a Dirichlet space (D.S.) for K , if

(1) $\forall f \in C(\overline{ex\ K})$ $\exists \tilde{f}^H \in H$: $\tilde{f}^H|_{\overline{exK}} = f$

(2) $f \geq 0 \Rightarrow \tilde{f}^H \geq 0$.

Example: K = unit ball in R^n.
$H = \{f \in C(K) : f$ is harmonic in the interior of $K\}$ is a D.S. for K.

For $f \in C(\overline{ex\ K})$ put
$D_f := \{g \in C(K) : g = \tilde{f}^H$ for some D.S. $H\}$
Clearly $D_f \subset [\underset{\smile}{f}, \hat{f}]$, the interval taken in C(K).

Theorem: (Mägerl, Papadopoulou, Cl.):
If K is stable and metrizable then there is a D.S. for K and moreover:

D_f is (uniformly) dense in $[\underset{\smile}{f}, \hat{f}]$
for all $f \in C(ex\ K)$.

Counterex. (Papadopoulou): Let K = unit ball in $\mathbb{R}^4$.
There is an $f \in C(ex\ K)$ such that

$$\underset{\smile}{f} \in C(K) \setminus D_f.$$

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