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Some remarks on Caratheodory construction of measures in metric spaces

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Some remarks on Caratheodory construction of measures in metric spaces

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If we have a metric space $\mathfrak{X} = (X,\varrho)$ and $A \subset X$, then by Halmos [2] p.53

$$H^p(A) := \sup_{\varepsilon>0} \inf\{\sum_{i=1}^{\infty} \delta^p(A_i) / A_i \in \mathcal{R}(X) \wedge \bigcup_{i=1}^{\infty} A_i \supset A \wedge \delta(A_i) \leq \varepsilon\}$$

is called the p-dimensional Hausdorff measure of A, where $p \in R_+ \setminus \{0\}$, $\mathcal{R}(X)$ is the set of all subsets of X and $\delta(B)$ denotes the diameter of $B \subset X$. General considerations on such a definition are given in the book of Federer [1] 169-171. I will start with these considerations.

Let $\mathcal{F}$ be a family of subsets of X and $\zeta : \mathcal{F} \longrightarrow \overline{R}_+ (= R_+ \cup \{\infty\})$ a function on $\mathcal{F}$. A sequence $(F_i)_{i\in N}$ is called an <u>allowed ε-covering</u> of A with respect to $\mathcal{F}$, iff

1. $F_i \in \mathcal{F}$ for all $i \in N$
2. $\bigcup_{i=1}^{\infty} F_i \supset A$
3. $\delta(F_i) \leq \varepsilon$.

If we define

$$i_\varepsilon(A) = \inf\{\sum_{i=1}^{\infty} \zeta(F_i) / (F_i)_{i\in N} \text{ is allowed } \varepsilon\text{-covering of } A \text{ with respect to } \mathcal{F}\}$$

so we obtain

a) $i_\varepsilon(A) \geq i_{\varepsilon'}(A)$ for $\varepsilon \leq \varepsilon'$

b) $i_\varepsilon(A \cup B) = i_\varepsilon(A) + i_\varepsilon(B)$ whenever $\varrho(A,B) > 2\varepsilon > 0$

The validity of a) is obviously.

To b): Let $(F_i^A)_{i\in N}$ and $(F_i^B)_{i\in N}$ be allowed ε-coverings of A,B respectively, then $F_1^A, F_1^B, F_2^A, F_2^B, \dots$ is an allowed ε-covering of $A\cup B$. Hence it holds

$$(*) \quad i_\varepsilon(A\cup B) \leq i_\varepsilon(A) + i_\varepsilon(B)$$

On the other hand let be $\varrho(A,B) > 2\varepsilon$, then every allowed ε-covering $(F_i^{A\cup B})_{i\in N}$ of $A\cup B$ consists of tow disjoint allowed ε-coverings $(F_i^A)_{i\in N}$ and $(F_i^B)_{i\in N}$ of A,B respectively, that means

$$(**) \quad i_\varepsilon(A\cup B) \geq i_\varepsilon(A) + i_\varepsilon(B) \text{ whenever } \varrho(A,B) > 2\varepsilon .$$

(*) and (**) is the proof for b).

a) implies

$$\psi(A) = \lim_{\varepsilon\to 0^+} i_\varepsilon(A) = \sup_{\varepsilon>0} i_\varepsilon(A) \text{ for all } A \subset X$$

$\psi : \mathcal{P}(X) \longrightarrow \bar{R}_+$ is an outer measure (i.e. $0 \le \psi(A) \le \infty$, $\psi(\emptyset) = 0$, $\psi(A) \le \psi(B)$ for $A \subset B$, $\psi(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \psi(A_i)$)

c) $\psi : \mathcal{P}(X) \longrightarrow \bar{R}_+$ is a metric outer measure,

i.e. $\psi(A \cup B) = \psi(A) + \psi(B)$ whenever $\varrho(A,B) > 0$

The proof of c) is a conclusion of b), namely $\varrho(A,B) > 0$ implies the existence of $\varepsilon_0 > 0$ such that $\varrho(A,B) > \varepsilon_0$. Then we get $i_\varepsilon(A \cup B) - i_\varepsilon(A) + i_\varepsilon(B)$ for all $\varepsilon < \frac{\varepsilon_0}{2}$, and that means $\psi(A \cup B) = \psi(A) + \psi(B)$ by definition of ψ.

Let us denote by $\mathcal{A}_\psi$ the σ-field of ψ-measurable sets ($A \subset X$ is called ψ-measurable, iff $\psi(E) = \psi(E \cap A) + \psi(E \cap A')$ for all $E \subset X$). For every metric outer measure $\phi : \mathcal{P}(X) \longrightarrow \bar{R}_+$ holds the following

Lemma: (Federer [1] p.75, Halmos [2] p.48)

Let $\phi : \mathcal{P}(X) \longrightarrow \bar{R}_+$ be an outer measure on X, then $\mathcal{A}_\phi \supset \mathcal{B}(X)$ iff ϕ is a metric outer measure, where $\mathcal{B}(X)$ denotes the σ-field of Borelsets of X.

By this lemma it holds $\mathcal{A}_\psi \supset \mathcal{B}(X)$. $\psi : \mathcal{A}_\psi \longrightarrow \bar{R}_+$ is called the Caratheodory measure on X with respect to $\mathcal{F}$ and $\zeta : \mathcal{F} \longrightarrow \bar{R}_+$.

Examples for Caratheodory measures:

1) $\zeta(F) := \delta(F)$ for all $F \in \mathcal{F}$

a) $\mathcal{F} - \{X\}$

If $|X| = 1$, then $\psi \equiv 0$. In the case $|X| > 1$ it holds $\psi(A) = \infty$ for all $A \subset X$

b) $\mathcal{F} = \{F / F \subset X \wedge |F| = 1\}$, then

$$\psi(A) = \begin{cases} 0 & \text{for } |A| \le \aleph_0 \\ \infty & \text{for } |A| > \aleph_0 \end{cases}$$

c) $X = \{x / x = 0 \vee x = \frac{1}{n}, n \in N\}$, $\varrho(a,b) = |a-b|$, $\mathcal{F} = \{F / F \subset X \wedge |F| \ge 2\}$

then $$\psi(A) = \begin{cases} 0 & \text{for } A = \{0\} \\ \infty & \text{otherwise} \end{cases}$$

d) $(X,\varrho) = (R,\varrho)$, $\mathcal{F} = \{F / F = [a,b[\wedge a,b \in R\}$, then ψ is the Lebesgue measure on R.

2) $\zeta(F) = \delta^p(F)$ for all $F \in \mathcal{F}$ and $p \in R_+ \setminus \{0\}$

a) $\mathcal{F} = \mathcal{P}(X)$, then ψ corresponds to the p-dim. Hausdorff measure H^p.

b) $\mathcal{F} = \{F / F$ is a closed ball in $X\}$, in this case ψ is called the p-dim. spherical measure over X.

The 1-dim. Hausdorff measure, the 1-dim. spherical measure and the set of between points

Let us start with the definition of betneen points

$x \in X$ is called between $a,b \in X$, $a \neq x$, $b \neq x$, iff

$$\rho(a,b) = \rho(a,x) + \rho(x,b) .$$

Let $B(a,b)$ be the set if all between points of $a,b \in X$ and $B^*(a,b) = B(a,b) \cup \{a,b\}$, then it holds for the reals

$$\rho(a,b) = \sigma(B^*(a,b)) = \lambda(B^*(a,b)) = H^1(B (a,b)) = S^1(B (a,b)),$$

where $a,b \in R$, ρ denotes the euclidian metric on the reals, λ the 1-dim. Lebesgue measure and S^1 the 1-dim. spherical measure. In my lecture in Warnemünde (in autumn 1977) "A special property of 1-dimensional Hausdorff measure". I asked for the validity of the equation

$\rho(a,b) = \sigma(B (a,b)) = H^1(B (a,b))$ in an arbitrary metric space X. The main result was the following

1 Theorem.

Let (X,ρ) be a complete and convex metric space (convex in he sense of Menger) and a,b X, hen the following conditions a e quivalent:

1 $\rho(a,b) - \sigma(B^*(a,b)) = H^1(B (,b))$

is pos i le to connec a d wi u i hort t rc.

3 B (a b) is n arc, i.e ho eo orphic to 0 1]

. nere is a unique metric segm nt $(a\ b)_s$ c nne ting a and b.

5 If $p,q \in B^*(,b)$ with $p \neq q$, then $p \in B^*(a\ q)$ o $\in B^*(q.b)$

Remarks:

1) An arc connecing $a,b \in X$ is a homeomorphism $f:[0,1] \xrightarrow[\text{into}]{} X$ such that $f(0)=a$ and $f(1)=b$.

A shortest arc connecting $a,b \in X$ is an arc $f:[0,1] \xrightarrow[\text{into}]{} X$ connecting a,b such that $l(f) \leq l(g)$ for all arcs $g:[0,1] \xrightarrow[\text{into}]{} X$ connecting a,b, where $l(f)$ denotes the length of the arc f.

2) $(a,b)_s$ is called a metric segm nt connecting a,b iff

1. $(a,b)_s \subset X$ and $a,b \in (a,b)_s$
2. $(a,b)_s$ is congruent to an interval $[x,y] \subset R$, i.e. there is an intervall $[x,y]$ and a metrical isomorphism $f:[x,y] \longrightarrow (a,b)_s$, such that $f(x)=a$ and $f(y)=b$.

Now let $(X,\| \|)$ be a n-dimensional normed vector space and

$$K_r(x) := \{y / y \in X \wedge \|x-y\| \leq r\}$$

the r-ball with the centre $x \in X$, then $K_r(x)$ is a convex and symmetrical set. A point $y \in K_r(x)$ is called an <u>extreme point</u> of $K_r(x)$, iff there is not a finite line $g \subset K_r(x)$, containing y in the relative interior of g. For example let us consider R^2 having the following two norms

1) $\|x\| := (x_1^2 + x_2^2)^{\frac{1}{2}}$, where $x = (x_1, x_2)$

   In this case every point $y \in FrK_r(x)$ is an extreme point.

2) $\|x\| := \sup\{|x_1|, |x_2|\}$

   Then $FrK_r(x)$ contains exactly four extreme points.

We obtain the following theorem as a conclusion of the theorem above.

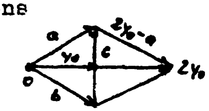
<u>2. Theorem:</u>

Let $(X, \| \|)$ be a n-dimensional normed vector space, then the following two conditions are equivalent:

1. $\varphi(x,y) = \sigma(B^*(x,y)) = H^1(B^*(x,y))$ for each $x, y \in X$
2. Every point y belonging to $FrK_r(x)$ is an extreme point of $K_r(x)$ for each $x \in X$ and $r > 0$.

<u>Proof</u>: $1 \Rightarrow 2$

We suppose $y_0 \in FrK_1(0)$ and y_0 is not extreme point of $K_1(0)$, hence there is a finite line $[a,b] \subset K_1(0)$, such that y_0 is the centre of $[a,b]$, i.e. $y_0 = \frac{1}{2}(a + b)$. The definition $c := a - y_0$ implies the following equations



$$\|a\| = \|y_0 + c\| = 1$$
$$\|b\| = \|y_0 - c\| = 1$$
$$\|2y_0 - a\| = \|2y_0 - y_0 - c\| = \|y_0 - c\| = 1 .$$

On the other hand it holds: $\|2y_0\| = 2\|y_0\| = 2$. Hence the arcs $[0,2y_0]$ and $[0,a] \cup [a,2y_0]$ are two shortest arcs connecting 0 and $2y_0$. That is a contradiction to theorem 1 condition 2. (If $y_0 \in FrK_r(x)$ and y_0 is not extreme point, so we get a contradiction in the same way)

$2 \Rightarrow 1$

On the supposition that 1. does not hold there are two points $x, y \in X$ and tow shortest arcs $f:[0,1] \longrightarrow X$, $g:[0,1] \longrightarrow X$ connecting x and y, such that $f \neq g$ and $l(f) = l(g) = \varphi(x,y)$. It is possible to find $a, b \in X$, such that

$$a \in g([0,1]) \wedge a \notin f([0,1])$$
$$b \notin g([0,1]) \wedge b \in f([0,1])$$
$$0 < \tilde{r} := l([x,a]_g) = l([x,b]_f) < \varphi(x,y),$$

where $l([x,a]_g)$ denotes the length of g from x to a and

$l([x,b]_f)$ denotes the length of f from x to b. We ask for the distance $\rho(x,y_0)$ and $\rho(y,y_0)$, where $y_0=\frac{1}{2}(a+b)$.

$$\rho(x,y_0)=|x-y_0|\leq\tfrac{1}{2}(|x-a|+|x-b|)=\tfrac{1}{2}(\rho(x,a)+\rho(x,b))=\tfrac{1}{2}(\tilde{r}+\tilde{r})=\tilde{r}$$

$$\rho(y,y_0)=|y-y_0|\leq\tfrac{1}{2}(\rho(y,a)+\rho(y,b))=\tfrac{1}{2}(\rho(x,y)-\tilde{r}+\rho(x,y)-\tilde{r})=$$
$$=\rho(x,y)-\tilde{r}$$

In the case $\rho(x,y_0)<\tilde{r}$ we obtain for the length of the arc $[x,y_0]\cup[y_0,y]$ connecting x and y

$$l([x,y_0]\cup[y_0,y])=l([x,y_0])+l([y_0,y])=\rho(x,y_0)+\rho(y_0,y)$$
$$<\tilde{r}+\rho(x,y)-\tilde{r}=\rho(x,y),\text{ i.e.}$$
$$l([x,y_0]\cup[y_0,y])<\rho(x,y).$$

That is impossible, hence $\rho(x,y_0)=\tilde{r}$. That means y_0 belongs to $\mathrm{Fr}K_{\tilde{r}}(x)$ and so we get a contradiction to 2.(because y_0 is not extreme point).

Now we consider some relations between the Hausdorff measure and the spherical measure on a metric space $\mathfrak{X}$. The following properties are well known (see Federer [1])

1. $H^p(A)\leq S^p(A)$ for all $A\subset X$ and $p\in R_+\setminus\{0\}$
2. If there is a real $c\geq 1$ for every subset $A\subset X$, such that A is contained in a closed ball having the diameter smaller or equal $c\cdot\delta(A)$, then $S^p(A)\leq c^p\,H^p(A)$. (c must be independent of A)

   $c=2$ fulfils the condition above. We obtain such a real number c in the n-dim. Euclidian space R^n by

<u>Jung's theorem</u>:(Federer 2.10.41)

If $A\subset R^n$ and $0<\delta(A)<\infty$, then A is contained in a unique closed ball with minimal diameter, which does not exceed

$$(2n/n+1)^{\frac{1}{2}}\cdot\delta(A).$$

For example if we consider a equilateral triangle Δ in $\mathbb{R}^2$, then the smallest closed ball containing Δ has the diameter $2r$, where $r=\frac{1}{\sqrt{3}}\cdot\delta(\Delta)$

3. If $A\subset X$ is congruent to a closed interval $[x,y]$, then $S^1(A)=|x-y|$.

Now it is easy to prove the following theorem.

<u>3. Theorem</u>

For every complete and convex metric space $\mathfrak{X}$ holds:

$$H^1(B^*(a,b))=\rho(a,b)\Longleftrightarrow S^1(B^*(a,b))=\rho(a,b)$$

for arbitrary $a,b\in X$.

<u>Proof:</u> „$\Rightarrow$"

By theorem 1 $H^1(B^*(a,b))=\rho(a,b)$ implies: $B^*(a,b)$ is congruent to a closed interval $[x,y]$ such that $\rho(a,b)=|x-y|$. Now we use property 3. and obtain $S^1(B^*(a,b))=|x-y|=\rho(a,b)$

$,\Leftarrow"$

By property 1. we have $H^1(B^*(a,b)) \leq S^1(B^*(a,b))=\rho(a,b)$. On the othe~ hand it holds $\rho(a,b) \leq H^1(B^*(a,b))$ in every complete and convex metric space.

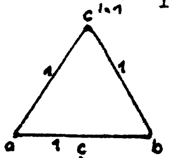
Remarks:

1) This theorem implies the validity of theorem 1 for the 1-dim. spherical measure.

2) By theorem 3 it holds $H^1(B^*(a,b))=\rho(a,b)$ implies $H^1(B^*(a,b))=S^1(B^*(a,b))$ for every complete and convex metric space. But in general it does not hold $H^1(A)=S^1(A)$ for $A\subset X$ and (X,ρ) complete and convex metric space. For example: Let (X,ρ) be the Euclidian plain R^2. We define

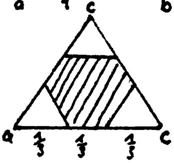
$$A := \bigcap_{i=1}^{\infty} A_i \quad , \text{ wnere}$$

A_1 =



set of all points belonging to the equilateral triangle

A_2 =



set of all points belonging to the smaller three eqilateral triangle

⋮ ⋮ ⋮

Then it holds: $H^1(A)=1$ and $S^1(A)=2/\sqrt{3}$

References:

[1] H. Federer : Geometric Measure Theory, Berlin 1969

[2] P. R. Halmos : Measure Theory, New York 1950

[3] W. Rinow : Die inner Geometrie der metrischen Räume, Berlin 1961