Siegfried Graf

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by
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Given a finite measure space $(X,\mathfrak{A},\mu)$, a measurable space $(Y,\mathfrak{B})$, and a $\mathfrak{B}$-$\mathfrak{A}$-measurable map $p: Y \to X$ let $\mathfrak{M}$ denote the set of all measures ν on $\mathfrak{B}$ with image measure $p(\nu)$ equal to μ. Then $\mathfrak{M}$ is convex and it is natural to ask the following questions:

1) What are the extreme points of $\mathfrak{M}$?
2) When is the set $\mathrm{ex}\,\mathfrak{M}$ of all extreme points of $\mathfrak{M}$ non empty and, therefore, $\mathfrak{M} \neq \emptyset$?
3) When does $\mathfrak{M}$ contain exactly one element ?

It is the aim of this talk to investigate these problems.

1. Characterization of extremal preimage measures

Problem 1 was first tackled by Douglas (1964) and later generalized by Plachky (1976) (see also Portenier (1974)). The following theorem is essentially a combination of the results due to these authors.

Theorem 1:

For a finite measure ν on $\mathfrak{B}$ the following statements are equivalent:

(i) $\nu \in \mathrm{ex}\,\mathfrak{M}$

(ii) There exists a σ-homomorphism $\Phi: \mathfrak{B} \to \mathfrak{A}/\mu$ with $\nu = \mu \circ \Phi$ and $A \in \Phi(p^{-1}(A))$ for all $A \in \mathfrak{A}$.

(iii) The map $f \mapsto \widetilde{f \circ p}$ ($f \in L^1(X,\mathfrak{A},\mu)$) is an isometry of $L^1(X,\mathfrak{A},\mu)$ onto $L^1(Y,\mathfrak{B},\nu)$.

(iv) For all $B \in \mathfrak{B}$ there is an $A \in \mathfrak{A}$ with $\nu(B \vartriangle p^{-1}(A)) = 0$.

Proof:

(i) $\Rightarrow$ (iii) is an easy consequence of the Hahn-Banach theorem (cf. Douglas (1964), Portenier (1974)).

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (ii): The map $\Phi: \mathfrak{B} \to \mathfrak{A}/\mu$ defined by $\Phi(B) = \tilde{A}$, where $A \in \mathfrak{A}$ with $\nu(B \vartriangle p^{-1}(A)) = 0$, is the σ-homomorphism we are looking for.

(ii) $\Rightarrow$ (i) is an easy consequence of the Radon-Nikodym theorem (cf. Graf (1977)).

Another characterization of extremal preimage measures was given by Edgar (1976) in the case that X and Y are compact spaces, μ is a

Radon measure and p a continuous map. In the rest of this section we shall be concerned with a generalization of Edgar's result. - Let us first restate a definition due to Edgar (1976).

Definition:
An $\mathfrak{A}_\mu$-$\mathfrak{B}$-measurable map f: $X \longrightarrow Y$ is called a measurable weak s ction for p iff $f^{-1}p^{-1}(A) \sim A$ for all $A \in \mathfrak{A}$.

It is clear that a y $\mathfrak{A}_\mu$-$\mathfrak{B}$-measurable section for p is also a measurable weak section for p. It follows immediately from the equivalence of (i) and (ii) in theorem 1 that for a measurable weak section f for p the image measure $f(\mu)$ is in ex$\mathfrak{M}$. In the special situation, described above, Edgar (1976) showed the converse, i.e. every extremal preimage measure, which is Radon, is the image of μ w.r.t. some measurable weak section for p. It is our aim to prove this converse under more general assumptions. The following lemma is the main step in this direction.

In what follows Y is always a Hausdorff space and $\mathfrak{B} = \mathfrak{B}(Y)$ the Borel σ-field of Y.

Definition:
A collection $\mathcal{K}$ of compact subsets of a Hausdorff space Z is called a base for the compact sets in Z iff $\mathcal{K}$ is stable w.r.t. finite intersections and iff every compact subset of Z is the intersection of sets from $\mathcal{K}$.

Lemma 1:
Let $(X,\mathfrak{A},\mu)$ be complete and Φ: $\mathfrak{B}(Y) \longrightarrow \mathfrak{A}/\mu$ a σ-homomorphism with $\mu\circ\Phi$ a Radon measure. Furthermore let F be a correspondence from X to Y such that there exists a lifting θ for $(X,\mathfrak{A},\mu)$, a sequence $(K_n)_n$ of compact subsets of Y, and, for every n, a base $\mathcal{K}_n$ for the compact subsets of K_n with the following properties:

(a) $\mu\circ\Phi$ is supported by $\bigcup_n K_n$.

(b) For all $x \in X$ and all n the set $F(x) \cap K_n$ is closed.

(c) There exists a μ-nullset N in X with $\theta\circ\Phi(K) \setminus N \subset F^{-1}(K)$ for all $K \in \bigcup_n \mathcal{K}_n$.

Then there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection f for F with $f^{-1}(B) \in \Phi(B)$ for all $B \in \mathfrak{B}(Y)$.

Proof:
For $x \in X$ define $\mathcal{F}_x = \{K \subset Y\colon K \text{ compact}, x \in \theta\circ\Phi(K)\}$ and $f\colon X \to Y$ by $f(x) \in \bigcap \mathcal{F}_x$, if $x \in \bigcup_n \theta\circ\Phi(K_n) \setminus N$ and $f(x) \in F(x)$ arbitrary

elsewhere. Then f has the desired properties.

As an immediate consequence of the lemma we get the next theorem

Theorem 2:

If $(X,\mathfrak{A},\mu)$, Y, and Φ are as in the lemma then there is an $\mathfrak{A}$ $\mathfrak{B}(Y)$-measurable map $f: X \to Y$ with $f^{-1}(B) \in \Phi(B)$ for all $B \in \mathfrak{B}(Y)$, i.e. Φ is induced by f.

The following corollary is a generalization of a result of Sikorski (1949).

Corollary:

If $(X,\mathfrak{A},\mu)$ is complete and Y a Radon space then every σ-homomorphism $\Phi: \mathfrak{B}(Y) \to \mathfrak{A}/\mu$ is induced by an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable map from X to Y.

Let us now return to our main object and state the generalized version of Edgar's theorem which follows from combining theorem 1, (i)$\Leftrightarrow$(ii), and theorem 2.

Theorem 3:

For a Radon measure ν on Y the following statements are equivalent:

(i) $\nu \in \text{ex}\,\mathfrak{M}$

(ii) There exists an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable weak section f for p with $\nu = f(\mu)$.

We shall now investigate under what conditions every element $\nu \in \text{ex}\,\mathfrak{M}$ is the image of μ w.r.t. some $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable section for p. The following corollary of theorem 3 gives a first answer.

Corollary:

If, in the situation of theorem 3, p is onto and there exists a countable family in $\mathfrak{A}$ which separates the points of X then the following are equivalent:

(i) $\nu \in \text{ex}\,\mathfrak{M}$

(ii) There exists an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable section f for p with $\nu = f(\mu)$.

To simplify the statem nts of the theorems we make the following definitions.

Definition:

Let ν be a finite Radon measure on Y and p: $Y \to X$ any map.

(a) p is called <u>ν-a.e. locally point-closed</u> iff there exists a sequence $(K_n)_n$ of compact subsets of Y such that ν is supported by $\bigcup_n K_n$ and $p^{-1}(x) \cap K_n$ is closed for all $x \in X$ and all n.

(b) A lifting θ for $(X,\mathfrak{A},\mu)$ is called <u>(p,ν)-almost strong</u> iff there exist a sequence $(K_n)_n$ of compact subsets of Y and, for every n, a base $\mathcal{K}_n$ for the compact subsets of K_n such that

(i) ν is supported by $\bigcup_n K_n$,

(ii) There is a μ-nullset N in X with $\theta(p(K)_*)\setminus N \subset p(K)$ for all $K \in \bigcup_n \mathcal{K}_n$.

(Here $p(K)_*$ is the complement of a measurable cover of $\complement p(K)$.)

Let us illustrate the preceding definitions by some examples.

(1) If $p^{-1}(x)$ is closed for all $x \in X$ then p is obviously ν-a.e. locally point-closed for all finite Radon measures ν on Y.

(2) If X is also a Hausdorff space and p is ν-Lusin measurable then p is ν-a.e. locally point-closed.

(3) If every compact subset of Y is metrizable then, for all $p: Y \longrightarrow X$ and all Radon measures ν on Y, there is a (p,ν)-almost strong lifting for $(X,\mathfrak{A}_\mu,\mu)$.

(4) If X is a Hausdorff space such that $(X,\mathfrak{A}_\mu,\mu)$ admits an almost strong lifting and if p is ν-Lusin measurable then there exists a (p,ν)-almost strong lifting for $(X,\mathfrak{A}_\mu,\mu)$.

Applying lemma 1 and theorem 1 we get

<u>Theorem 4:</u>

Let ν be a finite Radon measure on Y. If p is onto, ν-a.e. locally point-closed, and if there exists a (p,ν)-almost strong lifting for $(X,\mathfrak{A}_\mu,\mu)$ then the following are equivalent:

(i) $\nu \in \operatorname{ex}\mathfrak{M}$ and $K\setminus p^{-1}(p(K)_*)$ is a ν-nullset for all compact subsets K of Y.

(ii) There exists an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable section f for p with $\nu = f(\mu)$.

The condition $\nu(K\setminus p^{-1}(p(K)_*))=0$ for all compact sets $K \subset Y$ is satisfied if $p(K) \in \mathfrak{A}_\mu$ for these K. It is also satisfied if p is ν-Lusin measurable.

<u>Remark:</u>

It should be mentioned that under the assumptions of theorem 4 every Radon measure $\nu \in \mathfrak{M}$ with $\nu(K\setminus p^{-1}(p(K)_*))=0$ for all compact sets $K \subset Y$ has a strict disintegration w.r.t. p and μ.

2. Existence of extremal preimage measures

As we saw in section 1 the existence of measurable (weak) sections for p implies the existence of extremal preimage measures. We shall use this fact to prove the existence of extremal preimage measures.

a) Existence of measurable selections

The first theorem in this field was von Neumann's celebrated Measurable Choice Theorem (cf. von Neumann (1949)). This theorem was generalized by several authors (see for instance Aumann (1965), Sion (1960), Kuratowski - Ryll-Nardzewski (1965)). Using a theorem on continuous selections due to Hasumi (1969) we shall prove another selection theorem which is more suitable for our purposes.

Theorem 5:

Let X be a topological space, Y a regular Hausdorff space, $\mathfrak{B}(Y) \subset \mathfrak{A}$ and μ a finite measure on $\mathfrak{A}$ such that $(X,\mathfrak{A},\mu)$ is complete and admits a strong lifting. Furthermore let F be a point-compact upper semi-continuous correspondence from X to Y. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection f for F such that $f(\mu)$ is inner regular w.r.t. closed sets.

Proof:

For a strong lifting Θ for $(X,\mathfrak{A},\mu)$ the collection $\mathfrak{T}_\Theta = \{A \in \mathfrak{A}: A \subset \Theta(A)\}$ is an extremally disconnected topology on X which is stronger than the original topology. F is, therefore, an upper semi-continuous correspondence from $(X,\mathfrak{T}_\Theta)$ to Y. According to a selection theorem of Hasumi (1969) there is a $\mathfrak{T}_\Theta$-continuous selection f for F. It is easy to see that f has the desired properties.

b) Existence of measurable weak sections

Using the selection theorem we get the following

Theorem 6:

Let $(X,\mathfrak{A},\mu)$ be any finite measure space, Y a Hausdorff space, and $p: Y \to X$ a $\mathfrak{B}(Y)$-$\mathfrak{A}$-measurable map with

(1) $\mu(X) = \sup\{\mu^*(p(K)): K \subset Y \text{ compact}\}$

(2) $\mu^*(p(K)) = \inf\{\mu^*(p(U)): K \subset U,\ U \text{ open}\}$ for all compact $K \subset Y$

Then there exists an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable weak section f for p with $f(\mu)$ a Radon measure on Y.

Proof:

For reasons of simplicity let us assume that Y is compact, $F = p^{-1}$.

Let Θ be a lifting for $(X,\mathfrak{A}_\mu,\mu)$ and $\mathcal{T}_\Theta = \{A \in \mathfrak{A}_\mu : A \subset \Theta(A)\}$. Then Θ is a strong lifting for $(X,\mathfrak{A}_\mu,\mu)$ w.r.t. the topology $\mathcal{T}_\Theta$. Let $\overline{F}$ be the closure of F in $X \times Y$ w.r.t. the product topology. Condition (1) implies that $\overline{F}$ is a point-compact upper semi-continuous correspondence. Due to theorem 5 there is an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable selection f for $\overline{F}$ with $f(\mu)$ a Radon measure. From condition (2) we deduce that f is a weak section for p.

Corollary:

Provided p satisfies the assumptions of theorem 6 then there is a Radon measure in ex$\mathfrak{M}$.

Condition (2) in theorem 6 is, for instance, satisfied in the following cases:

(i) X is a Hausdorff space, μ a Radon measure on X and $p: Y \to X$ is continuous.

(ii) Y is normal, every compact subset of Y is a G_δ-set, $p^{-1}(x)$ is compact for all $x \in X$, and $p(U) \in \mathfrak{A}_\mu$ for all open $U \subset Y$ or $p(F) \in \mathfrak{A}_\mu$ for all closed $F \subset Y$.

Combining this last fact with theorem 6 and theorem 4 we get the following

Proposition:

Let $(X,\mathfrak{A},\mu)$ be any finite measure space, Y a Hausdorff space in which every compact subset is metrizable, and $p: Y \to X$ a $\mathfrak{B}(Y)$-$\mathfrak{A}$-measurable onto map. Assume there is a Radon measure ν on Y with $p(\nu) = \mu$ and p ν-a.e. locally point-closed and $p(K) \in \mathfrak{A}_\mu$ for all compact $K \subset Y$. Then there is an $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable section for p.

3. Uniqueness of preimage measures

Again Y is always a Hausdorff space, while $p: Y \to X$ is a $\mathfrak{B}(Y)$-$\mathfrak{A}$-measurable map, $\mathfrak{M} = \{\nu : \nu$ measure on $\mathfrak{B}(Y)$ with $p(\nu) = \mu\}$.

Definition:

Two $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$ measurable maps $f_1, f_2: X \to Y$ are called <u>weakly μ-equivalent</u> iff $\mu(f_1^{-1}(B) \quad f_2^{-1}(B)) = 0$ for all $B \in \mathfrak{B}(Y)$.

Theorem 7:

Under the assumptions of theorem 6 the following statements are equivalent.

(i) $\mathfrak{M}$ contains exactly one Radon measure.

(ii) Any two $\mathfrak{A}_\mu$-$\mathfrak{B}(Y)$-measurable weak sections f_1, f_2 for p with $f_1(\mu)$, $f_2(\mu)$ Radon measures on Y are weakly μ-equivalent.

(iii) For disjoint compact sets $K_1, K_2 \subset Y$ the equality

$$\mu^*(p(K_1) \cup p(K_2)) = \mu^*(p(K_1)) + \mu^*(p(K_2))$$

holds.

If Y is σ-compact and metrizable then (i)-(iii) are equivalent to

(iv) $\mu^*(\{x \in X: \operatorname{card}(p^{-1}(x)) \geq 2\}) = 0$ and $p(K) \in \mathfrak{A}_\mu \cap p(Y)$ for all compact $K \subset Y$.

The equivalence of (i) and (iv) has been proved by Eisele (1975) for a special case and by Lehn-Mägerl (1977) in a different situation. For other considerations concerning the uniqueness of preimage measures we refer to Yershov (1974).

A more detailed account on the subject of the two last sections of this talk can be found in Graf (1977).

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Siegfried Graf

Mathematisches Institut
der Universität Erlangen-Nürnberg

Bismarckstr. 1 1/2

8520 Erlangen

Federal Republic of Germany