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ON STRONG MIXING AND WEAK CONVERGENCE FOR GROUPS OF OPERATORS

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§1

Let (X, Σ, μ) be a probability space and $\tau: X \longrightarrow X$ a measure preserving, invertible, bi-measurable transformation. Then τ induces an invertible isometry T of $L^p(\mu)$, $1 \leq p < \infty$, given as $Tf = f \circ \tau$.

We recall that τ is said to be ergodic if $\tau^{-1}(A) = A$ (up to a null set) implies $\mu(A) = 0$ or 1, and τ is called strongly mixing if for all $A, B \in \Sigma$ we have $\lim \mu(\tau^n(A) \cap B) = \mu(A)\mu(B)$. Clearly, every strongly mixing transformation is ergodic (put A = B). Two standard examples of ergodic and strongly mixing transformations are, respectively, the irrational rotation of the torus and the bilateral Bernoulli shift.

In terms of convergence, the ergodic transformations are characterized by the classical mean ergodic theorem which asserts that, given $1 \leq p < \infty$, the transformation τ is ergodic iff

$$\lim \frac{1}{n} \sum_{i=1}^{n} T^i f = \int f \, d\mu$$

in L^p-norm for every $f \in L^p(\mu)$.

An analogous characterization of strongly mixing transformations was found by Blum and Hanson [4]:

Given $1 \leq p < \infty$, the transformation τ is strongly mixing iff

$$\lim \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f = \int f \, d\mu$$

in L^p-norm for every $f \in L^p(\mu)$ and every strictly increasing sequence (k_i) of integers.

This last theorem has been generalized by many authors in various directions. Some more recent papers deal with cyclic and one-parameter semigroups of contractions in L^p-spaces ([2], [3], [5], [6], [9], [10]) and with cyclic order contractive semigroups in Banach lattices with order continuous norm ([8], see also [11], V §8).

§2

Here we present an extension of the Blum - Hanson theorem in another direction. We consider an arbitrary locally compact non-compact group G and a weakly measurable uniformly bounded representation of G on a Banach space E, say, $E = L^p(\mu)$. The property of strong mixing is now replaced by the condition: $\lim \langle T_t x, y \rangle = \langle x, y \rangle$ $(t \to \infty,\ x \in E,\ y \in E')$ or, more generally, by the existence of the weak operator limit: $\lim T_t = P$.

Also the averaging by means of increasing sequences (k_i) requires a modification (and in fact can be generalized to a wider class of averaging procedures) in the present situation. Following a definition introduced by Fong in [5], we denote by $\mathfrak{A}$ the family of all sequences (μ_n) of signed Radon measures on G satisfying:

(1) $\sup_n \|\mu_n\| < \infty$,

(2) $\lim_n \mu_n(G) = 1$,

(3) $\lim_n \sup_{t \in G} |\mu_n| (tK) = 0$ for every compact $K \subset G$.

Let us note that the sequence $\left(\frac{1}{n} \sum_{i=1}^{n} \delta_{k_i} \right)$ appearing in the classical Blum- Hanson theorem is in $\mathfrak{A}$, with $G = Z$.

Another simple example of $(\mu_n) \in \mathfrak{A}$ can be obtained by letting $G = R$ and μ_n be a normal distribution with dispertion σ_n^2, where $\sigma_n \to \infty$ as $n \to \infty$.

Now, instead of the finite sums $\frac{1}{n}\sum T^{k_i}$ we have integrals $\int T_t \, d\mu_n(t) \in \mathcal{L}(E)$ with $(\mu_n) \in \mathfrak{A}$. These operator integrals are understood in the sense specified below.

Let ν be a bounded signed Radon measure on G and suppose the orbit $O_x = \{T_t x : t \in G\}$ of every element x from a dense subspace E_0 of E is relatively weakly compact. Then for each $x \in E_0$ the mapping $y \to \int \langle T_t x , y \rangle \, d\nu(t)$ defines a bounded linear functional $\int T_t x \, d\nu(t)$ on E'. By the compactness of O_x and the Krein- Šmulian theorem, the closed convex hull of O_x is weakly compact, so that the above functional is in fact in E. Now, by the uniform boundedness of T_t, $t \in G$, the mapping $x \to \int T_t x \, d\nu(t)$ defines a bounded linear operator $E_0 \to E$ which clearly extends uniquely to an operator

$$\int T_t \, d\nu(t) \in \mathcal{L}(E).$$

Obviously, if ν has finite support or if E is reflexive, then $\int T_t x \, d\nu(t)$ $(x \in E)$ and $\int T_t \, d\nu(t)$ exist.

§3

Now we are in a position to state our main results. In the proofs we have made an extensive use of methods developed by Blum and Hanson, Akcoglu, Sucheston, Fong, and Nagel.

First we consider the case of E being a Hilbert space.

1. Let G be a locally compact non-compact group and let $t \to T_t$ be a weakly measurable unitary representation of G

on a complex Hilbert space H. Then for each x H the following conditions are equivalent:

(i) $\text{weak-}\lim_{t\to\infty} T_t x = x_o$,

(ii) $\lim_n \int T_t x \, d\mu_n(t) = x_o$ for every $(\mu_n) \in \mathfrak{A}$,

(iii) $\text{weak-}\lim_n \int T_t x \, d\mu_n(t) = x_o$ for every $(\mu_n) \in \mathfrak{A}$ such that the μ_n's are probability measures with finite supports.

The above theorem combined with the fact that on order intervals determined by functions from $L^\infty(\mu)$ the L^p-topologies for all $1 \le p < \infty$ coincide, we obtain the following extension of the Blum - Hanson theorem:

2. Let a locally compact non-compact group G act measurably on a probability space (X, Σ, μ) by $t \longrightarrow \tau_t$ and let $T_t f = f \circ \tau_t^{-1}$ be the isometric representation of G in $L^p(\mu)$, $1 \le p < \infty$. Then the following conditions are equivalent:

(o) $\lim_{t\to\infty} \mu(\tau_t(A) \wedge B) = \mu(A)\,\mu(B)$ for all $A, B \in \Sigma$ (i.e. the action of G is a "strongly mixing flow"),

(i) $\lim_{t\to\infty} T_t = \mu \otimes 1$ in the weak operator topology,

(ii) $\lim_n \int T_t \, d\mu_n(t) = \mu \otimes 1$ in the strong operator topology for every $(\mu_n) \in \mathfrak{A}$,

(iii) $\lim_n \int T_t \, d\mu_n(t) = \mu \otimes 1$ in the weak operator topology for every $(\mu_n) \in \mathfrak{A}$ such that the μ_n's are probability measures with finite supports.

Next, by using Banach lattice technique, the above result can be extended as in [8] (see also [11], V §8) to order contractive representations of G on arbitrary Banach lattice with order continuous norm. Finally, using Nagel's method (developed originally for cyclic semigroups) and applying the Kakutani fixed point theorem as well as the Bartle- Dunford-Schwartz representation of relatively weakly compact sets in ca(Σ), we obtain the following result, corresponding to a theorem of Akcoglu and Sucheston [1]:

3. Let E be a Lebesgue space and let $t \longrightarrow T_t$ be a weakly continuous isometric representation of a locally compact non-compact group G on E. Then the following conditions are equivalent:

(i) $\lim_{t\to\infty} T_t = P$ for the weak operator topology,

(ii) for any $(\mu_n)\in\mathfrak{A}$ the integrals $\int T_t \, d\mu_n(t)$ exist and converge to P for the strong operator topology,

(iii) $\lim_n \int T_t \, d\mu_n(t) = P$ in the weak operator topology for every $(\mu_n)\in\mathfrak{A}$ for which all μ_n are probability measures with finite supports.

For more details and the proofs of the above results we refer to [7].

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