Jan Kisyński

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Let R be a strongly continuous representation of a Lie group G on a real or complex Hilbert space H , $D_\infty(R)$ the set of all C^∞- vectors of R , and dR the differential of R . Let $X_0, X_1, \dots, X_N$ be a set of left-invariant vector fields on G . Let $A = (a_{mn})$ be $N \times N$ - matrix, hermitian positive definite or symmetric positive definite, depending of whether the space H is complex or real. Consider the left invariant differential operator

$$P = \sum_{m,n=1}^{N} a_{mn} X_m X_n + \sum_{n=1}^{N} c_n X_n + X_0$$

and let dR(P) be its image by the differential of R . The domain of dR(P) is $D_\infty(R)$, by definition.

THEOREM 1. Under the above assumptions dR(P) is a pregenerator of a one-parameter strongly continuous $\mathcal{L}(H)$-valued semigroup. If H is a complex Hilbert space and $X_0 = \sum_{m,n=1}^{N} b_{mn}[X_m, X_n]$ with $b_{mn} \in \mathbb{R}$, then that semigroup has holomorphic extension onto a sector $\{t : 0 \neq t \in \mathbb{C}, |\operatorname{Arg} t| < \alpha\}$ with some $\alpha \in (0, \frac{1}{2}\pi]$. If $X_0 = 0$ and the space H is either real or complex, then dR(P) is a pregenerator of a strongly continuous $\mathcal{L}(H)$-valued cosine function.

When the representation R is unitary, $X_0 = 0$, and the coefficients c_n are purely imaginary, then d R(P) is symmetric and semi-bounded from above. The property that dR(P) is a pregenerator is then equivalent to essential selfadjointness of dR(P). Thus our theorem implies the result of P.Jørgensen [Journal of Functional Analysis 20 (1975), Corollary 1.1, p.113]. Let us indicate that while Jørgensen's proof is based on a strong result of Hörmander about hypoellipticity of some second order differential operators, our method uses only some abstract semi-group theory and elementary integration by parts of the type of L.Gårding [Bull. Soc. Math. France 88 (1960), p. 73-93].

As an extra product we are able to give a purely analytical proof of an estimation for convolution semigroups of probability measures on non-compact Lie groups, which generalizes and reinforces the result proved by E.Nelson [Annals of Math. 70 (1959), p. 572-615, lemma 8.1] by advanced probabilistic methods. Namely, let p_t , $t \geq 0$, be a convolution semigroup of probability measures on G We shall say that p_t is a Lindeberg semigroup iff

$$p_t(U) = 1 - o(t) \quad \text{as} \quad t \downarrow 0 \quad ,$$

for any open neighbourhood U of the neutral element e of G. / To any convolution semigroup p_t of probability measures on G there corresponds a G-valued Markov process with transition probabilities $P_t(x,E)= p_t(x^{-1}E)$, and it can be proved that /almost/ all trajectories of this process are continuous iff the semigroup p_t is Lindeberg./ Let $C_0(G)$ be the space of all continuous functions on G having limit 0 at infinity. To convolution semigroup p_t one can associate the $\mathcal{L}(C_0(G))$ -valued semigroup $S(t)$ defined by

$$[S(t)\varphi](x) = \int_G \varphi(xy)\, p_t(dy) \ , \quad \varphi \in C_0(G) \ .$$

If p_t is Lindeberg then according to G.A.Hunt [Trans. Amar. Math. Soc. 81 (1956) , p.264-293 , section 3] there are left-invariant vector fields $X_0, X_1, \ldots, X_N$ on G such that $\lim_{t\downarrow 0} \frac{1}{t}\{[S(t)\varphi](x)-\varphi(x)\} = \left(\sum_{n=1}^{N} X_n^2 + X_0\right)\varphi(x)$ for every $\varphi \in C_0(G)$. Let $\tau(x)$ be the distance from e to x in the sense of an arbitrarily fixed left-invariant Riemann metric on G and put

$$a = \lim_{\xi\downarrow 0} \xi^{-2} \sum_{n=1}^{N} [\tau(\exp \xi X_n)]^2 \ .$$

THEOREM 2. With the above notations for any Lindeberg convolution semigroup p_t , $t\geq 0$, of probability measures on G we have

$$\sup_{0<t\leq T} \int_G \exp\left[\frac{[\tau(x)]^2}{4a\,(t+\varepsilon)}\right] p_t\,(dx) < \infty$$

for every finite $T>0$ and every $\varepsilon>0$.

To illustrate this result, let $G = \mathbb{R}$, $\tau(x) = |x|$ and $p_t(E)=\frac{1}{2}(\pi t)^{-1/2} \int_E \exp\left[-\frac{(x-vt)^2}{4t}\right]dx$, which corresponds to diffusion with simultaneous convection. Then $a = 1$ and $\int_{\mathbb{R}} \exp\left[\frac{x^2}{4(t+\varepsilon)}\right] p_t\,(dx) = \left(1+\frac{t}{\varepsilon}\right)^{1/2} \exp\left[\frac{v^2t^2}{4\varepsilon}\right]$.