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AUTOMATIC CONTINUITY OF TRANSLATION-INVARIANT LINEAR OPERATORS

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The present survey is based on some recent results from a forthcoming joint paper with Ernst Albrecht. We are concerned with the problem of continuity for certain linear operators between spaces of functions and distributions. For the sake of illustration, let us consider the following typical situation from the physics of linear systems. Let $\Theta:\mathcal{D}(\mathbb{R})\to\mathcal{D}'(\mathbb{R})$ denote a linear operator which is causal and translation-invariant in the following sense:

(C) $f=o$ on $]-\infty,t[$ implies $\Theta f=o$ on $]-\infty,t[$, for all $f\in\mathcal{D}(\mathbb{R})$.

(T) $\Theta\circ T_\alpha = T_\alpha\circ\Theta$ for all $\alpha\in\mathbb{R}$.

Here T_α stands for translation by α to the right, i.e. $T_\alpha f(t) :=f(t-\alpha)$ for all $t\in\mathbb{R}$, resp. $\langle T_\alpha\varphi,f\rangle:=\langle\varphi,T_{-\alpha}f\rangle$ for all $f\in\mathcal{D}(\mathbb{R})$ and $\varphi\in\mathcal{D}'(\mathbb{R})$. Assume in addition that the operator $\Theta:\mathcal{D}(\mathbb{R})\to\mathcal{D}_b'(\mathbb{R})$ is continuous. Then, as a well-known consequence of the Laurent Schwartz kernel theorem, there exists some distribution $S\in\mathcal{D}'(\mathbb{R})$ with $\operatorname{supp} S\subset[o,\infty[$ such that $\Theta f=S*f$ for all $f\in\mathcal{D}(\mathbb{R})$. This result is of fundamental importance in the theory of linear systems. However, in contrast to the conditions (C) and (T) which are quite natural and accessible to physical experiments, the continuity assumption on Θ is by no means satisfactory. There has been some considerable effort to drop this additional assumption in the special case of certain passive linear systems; but dissipativity assumptions are rather strong, and peculiar techniques are available when they prevail. Thus it is somewhat surprising that the following positive result holds without any restriction.

1. Theorem. Every linear operator $\Theta:\mathcal{D}(\mathbb{R})\to\mathcal{D}_b'(\mathbb{R})$ satisfying (C) and (T) is continuous and hence a convolution operator.

This theorem, as well as a number of similar results concerning different spaces of (even vector-valued) functions and distributions, is just one achievement of a far-reaching new theory on automatic continuity. In the sequel, some central features of this theory will be sketched in appropriate specialization.

2. Theorem. Consider a sequence $(X_n)_{n=0,1,\ldots}$ of F-spaces X_n and a sequence $(T_n)_{n=1,2,\ldots}$ of continuous linear operators $T_n:X_n\to X_{n-1}$. Let $(Y_n)_{n=0,1,\ldots}$ be a sequence of topological vector spaces Y_n and let $(\pi_n)_{n=1,2,\ldots}$ be a sequence of bounded linear operators $\pi_n:Y_0\to Y_n$. Assume that Y_0 can be represented as the union of some countable family of bounded subsets. Now, let $\Theta:X_0\to Y_0$ denote a linear operator such that $\pi_n\Theta T_1\cdots T_n: X_n\to Y_n$ is continuous for all $n=1,2,\ldots$. Then there exists some n such that $\pi_k\Theta T_1\cdots T_n:X_n\to Y_k$ is continuous for all $k=1,2,\ldots$.

The preceding result is based on a refined gliding hump technique and applies not only to the theory in question, but also to the problem of continuity for homomorphisms, derivations, intertwining operators, local operators, etc. One of the applications (proved via product spaces) is the following extended version of a uniform boundedness type theorem due to Vlastimil Pták. Note that corollary 3 in particular applies to linear operators from F-spaces to DF-spaces; on the other hand, the assertion does not hold in general for linear operators from Banach spaces to Fréchet spaces.

3. Corollary. Consider a pointwise bounded family $\{\Theta_\alpha:\alpha\in I\}$ of linear operators $\Theta_\alpha:X\to Y$ from an F-space X to a topological vector space Y which has a fundamental sequence of bounded subsets. Assume that each operator Θ_α is continuous on some closed

subspace X_α of X. Then $\{\theta_\alpha:\alpha\in I\}$ is equicontinuous on some finite intersection of the spaces X_α.

A slight generalization of theorem 2 leads to the following result which in particular confirms the automatic continuity of those causal linear operators commuting with at least one shift. Note, however, that some boundedness condition on the range space is needed. And observe that the scope of theorem 4 can be expanded by combining it with suitable closed graph theorems.

4. Theorem. Let $T:A\to L(X)$ denote a homomorphism from some semigroup $(A,\circ)$ to the semigroup of all continuous linear operators on some F-space X. And let Y be a vector space which is the union of some countable family of bounded subsets with respect to the projective topology given by a system $(Y_\alpha,\pi_\alpha)_{\alpha\in A}$ of topological vector spaces Y_α and linear mappings $\pi_\alpha:Y\to Y_\alpha$. Then any linear operator $\theta:X\to Y$ with the following two properties is continuous.

($\tilde{C}$) $\pi_\beta\theta T_{\beta\circ\alpha}:X\to Y_\beta$ is continuous for all $\alpha,\beta\in A$.

($\tilde{T}$) For all $\alpha,\beta\in A$ there exists a continuous linear operator $U_{\alpha\beta}:\pi_{\alpha\circ\beta}\theta T_\alpha(X)\to Y_\beta$ such that $\pi_\beta\theta=U_{\alpha\beta}\pi_{\alpha\circ\beta}\theta T_\alpha$.

5. Example. Let X denote the space of all $f\in CB^m(\mathbb{R})$, $C^m(\mathbb{R})$, $L^p(\mathbb{R})$, $L^p_{loc}(\mathbb{R})$, or $\mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} f\subset [o,\infty[$, where $m=o,1,\dots,\infty$ and $o<p\leq\infty$. And let Y denote the space of all $f\in B(\mathbb{R})$, $CB^m(\mathbb{R})$, $L^p(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, or $C^mV(\mathbb{R})$ such that $\operatorname{supp} f\subset [o,\infty[$, where $m=o,1,\dots,\infty$, $o<p\leq\infty$ and $V=(v_n)_n$ is a decreasing sequence of continuous weight functions $v_n:\mathbb{R}\to]o,\infty[$. Endow X and Y with the respective natural topologies. Then each linear operator $\theta:X\to Y$ satisfying $\theta\circ T_\alpha=T_\alpha\circ\theta$ for at least one $\alpha>o$ is continuous.

In contrast let us mention that the last assertion ceases to be true if Y is chosen as the space of all $f\in C(\mathbb{R})$ such that $\operatorname{supp} f\subset [o,\infty[$. And there are examples of discontinuous causal linear operators $\theta:\mathcal{D}(\mathbb{R})\to CB(\mathbb{R})$ which satisfy $\theta\circ T_\alpha=T_\alpha\circ\theta$ for some $\alpha>o$. Thus the situation becomes more complicated if spaces of a different type are involved. Nevertheless, the problem of continuity can be attacked in a similar, though somewhat more sophisticated fashion. We conclude with the following general principle from which theorem 1 can be easily derived. Another consequence of theorem 6 is the automatic continuity of all causal and translation-invariant linear operators $O: \mathcal{E}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$.

6. Theorem. Let $X = \operatorname{ind}_{\alpha\to\infty} X_\alpha$ be the inductive limit of an inductive spectrum $\{X_\alpha, \subset\}_{\alpha>0}$ consisting of F-spaces X_α in X with each connecting map being inclusion. And let $(T_\alpha)_{\alpha>0}$ denote a semigroup of linear operators $T_\alpha: X\to X$ such that $T_\alpha(X_\beta) \subset X_{\alpha+\beta}$ and such that the restriction $T_\alpha | X_\beta : X_\beta \to X_{\alpha+\beta}$ is continuous for all $\alpha,\beta>0$. Finally, let $Y = \operatorname{proj}_{\infty\leftarrow\alpha} Y_\alpha$ be the projective limit of a projective spectrum $\{Y_\alpha, \pi\}_{\alpha>0}$ consisting of topological vector spaces Y_α, each of them being the union of some countable family of bounded sets. Then any linear operator $\Theta: X\to Y$ with the following two properties is continuous.

($\tilde{C}$) $\pi_\beta \Theta T_{\beta+\alpha}: X\to Y_\beta$ is continuous for all $\alpha,\beta>0$.

($\tilde{T}$) For all $\alpha,\beta>0$ there exist a linear subspace $Z_{\alpha\beta}$ of $Y_{\alpha+\beta}$ and a topological isomorphism $U_{\alpha\beta}: Z_{\alpha\beta}\to Y_\beta$ such that $\pi_{\alpha+\beta}\Theta T_\alpha(X) \subset Z_{\alpha\beta}$ and $\pi_\beta \Theta = U_{\alpha\beta}\pi_{\alpha+\beta}\Theta T_\alpha$.