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Positivity Properties of Measures in Euclidean Quantum Field Theory

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A Wightman state consists of a sequence $W=(W_0, W_1, \ldots)$ of tempered distributions $W_n \in \mathcal{S}(\mathbb{R}^{4n})'$, satisfying several conditions which follow from general physical principles. These properties of the W_n imply that they are in fact boundary values of functions $\hat{W}_n$, which are analytic in certain domains in $\mathbb{C}^{4n}$. Among the points of analyticity are the so called Euclidean points $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^{4n}$, $z_\nu = (ix_\nu^0, \vec{x}_\nu)$, $(x_0, \vec{x}_\nu) =: x_\nu \in \mathbb{R}^4$, (i.e. the time component of z_ν is purely imaginary and the space components real), and $z_\mu \neq z_\nu$ for $\mu \neq \nu$. The functions $S_n(x_1, \ldots, x_n) = \hat{W}_n(z_1, \ldots, z_n)$ are called Schwinger functions. Lorentz invariance of the W_n implies that these functions are invariant w.r.t. the group of Euclidean motions in $\mathbb{R}^4$. Moreover, they are totally symmetric under permutations of the x_ν.

Models of quantum field theory are usually obtained by constructing the Schwinger functions. An important tool has been the theory of measures on infinite dimensional vector spaces: One represents the S_n as moments of a measure μ on the space of real tempered distributions $\mathcal{S}_R(\mathbb{R}^4)'$ in the sense that

$$\int_{\mathbb{R}^{4n}} S_n(x_1, \ldots, x_n)\, f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = \int_{\mathcal{S}'_R} \langle \omega, f_1 \rangle \cdots \langle \omega, f_n \rangle \, d\mu(\omega) \quad (*)$$

for all $f_1, \ldots, f_n$ in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. (Remark: Since the S_n are only defined if $x_\nu \neq x_\mu$, the left hand side is not defined unless the supports of the f are disjoint. In all known models, however, the right hand side defines an extension of the S_n as distributions to all points.)

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In order that the distributions S_n defined by the right hand side of (*) should lead back to a Wightman theory, they must satisfy a condition that was first explicitly stated by Osterwalder and Schrader and follows from the fact that a Wightman functional is a positive linear functional on the test function algebra $\underline{\mathcal{S}}$.
A strengthened version of the Osterwalder - Schrader condition is the following requirement for the measure (reflection positivity or T-positivity[7]):

$$\int \overline{f(\theta\omega)}\, f(\omega)\, d\mu(\omega) \geqslant 0 \qquad (**)$$

for all functions on $\mathcal{S}'_R$ of the form $f(\omega) = g(\langle\omega, f_1\rangle, \ldots, \langle\omega, f_n\rangle)$, where g is a bounded continuous function on $\mathbb{C}^n$ and $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^4)$ with support in $\{x \mid x^\circ \geqslant 0\}$. Here θ denotes reflection of the time coordinate x°, considered as an operation on $\mathcal{S}'$.

In the models studied so far it turns out that μ can be taken to be a positive measure, i.e.

$$\int f(\omega)\, d\mu(\omega) \geqslant 0$$

for all $f \geqslant 0$. This property is at first sight rather surprising, for it does not seem to have anything to do with the positivity of the Wightman states. But it is this property which is the basis of the close formal analogy between quantum field theory and statistical mechanics. It is therefore of some interest to understand this property better. In this connection the following has been shown to hold [9]:

<u>Proposition:</u> Let μ be a finite, complex Borel measure on $\mathcal{S}'_R$. Assume that μ is invariant under the action of the time-translation group on $\mathcal{S}'_R$ and satisfies (**). Then μ is a positive measure.

The problem of proving the existence of a representation by a measure for the Schwinger functions of a general Wightman theory is much more difficult. Some results in this direction have been obtained in [1-8] and may be summarized as follows:
1. The existence of a positive representing measure is connected with the problem of extending the Schwinger functions as distributions to all points in such a way that a certain positivity condition is fulfilled.
2. If this extension preserves invariance under space-time translations, then the measure can also be chosen to be invariant.

3. A representation by a complex measure is possible if and only if the S_n satisfy certain conditions on their behaviour near points where two or more of the x_ν coincide.
4. There are cases, where the Schwinger functions have a representation by an invariant complex measure, but no representing measure satisfies reflection positivity.

The problems of finding general criteria for a representation by a complex, invariant measure and a measure satisfying (**) are still open.

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