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# INDEPENDENT FAMILIES ON COMPLETE BOOLEAN ALGEBRAS

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We present definitions and lemmas concerning a proof of the following fact, without any set-theoretical assumptions.

Theorem. Every infinite complete Boolean algebra contains a free subalgebra of the same cardinality.

This solves the Question 44 of [vD,M,R]. The history of this problem and a survey of partial solutions ([Ko],[Ky],[M]) is given in [Bla].

The theorem extends the classical result of Hausdorff and Pospíšil concerning complete atomic BA's $(= \mathcal{P}(K))$ to arbitrary cBA's.

Let us summarize some well-known consequences of the Theorem. In what follows, B denotes an infinite cBA and X denotes an infinite extremally disconnected compact (e.d.c.) space.

C1 Let $\mathcal{U}(B)$ be the set of all ultrafilters on B, then $\mathrm{card}(\mathcal{U}(B)) = 2^{\mathrm{card}(B)}$; equivalently, $\mathrm{card}(X) = 2^{w(X)}$, where $w(X)$ is the weight of X.

C2 There are many $(= |\mathcal{U}(B)|)$ ultrafilters on B which have the character (= the least cardinality of a set of generators) equal to $|B|$.

The consequences C1 and C2 solve problems raised by Efimov [Ef].

C3 If C is a cBA with $|C| \leq |B|$ then there is a homomorfism $f : B \xrightarrow{\text{onto}} C$; equivalently, for an e.d.c. space Y with $w(Y) \leq w(X)$ there is an embedding of Y into X.

C4 There is a continuous mapping $f : X \xrightarrow{\text{onto}} \{0,1\}^{w(X)}$ .

C5 The space $X$ contains a copy of itself as a nowhere dense subset and therefore $X$ is not homogeneous. [F].

Notations, definitions

For a BA $B$ let $B^+ = B - \{0\}$. For $u \in B^+$ let $B_u$ denote a "partial subalgebra" of $B$ with the universe $\{v \leq u ; v \in B\}$.

(i) Part $(B) = \{p \subseteq B^+ ; \bigvee p = 1$ and the elements of $p$ are pairwise disjoint$\}$ .

(ii) $\mathcal{P} \subseteq$ Part $(B)$ is called an independent family of partitions if for any finite set of partitions $\{p_0, \ldots, p_{n-1}\} \subseteq \mathcal{P}$ and every mapping $f : n \longrightarrow \bigcup \{p_i , i<n\}$ with $f(i) \in p_i$ we have $\bigwedge \{f(i) , i<n\} \neq 0$ .

(iii) $B$ is semifree if there is an independent family of partitions $\mathcal{P}$ on $B$ with $|\mathcal{P}| = |B|$ .

Hence the theorem is equivalent to the statement "every infinite cBA is semifree" .

(iv) $D \subseteq B^+$ is dense in $B$ if $(\forall v \in B^+)(\exists u \in D)\ u \leq v$ ;

$d(B) = \min \{\text{card } (D) ; D$ is dense in $B\}$ .

(v) sat $(B) = \min \{\nu ; (\forall p \in \text{Part } (B)) (|p| < \nu)\}$ (! less than)

Trivially, sat $(B) \geq$ sat $(B_u)$ , $d(B) \geq d(B_u)$ for $u \in B^+$ . Hence for a cBA $B$ there is a partition $p$ such that

$B = \sum_{u \in p} B_u$ (a product in the category of BA's ) and all $B_u$'s are homogeneous in sat and $d$ .

(vi) (Erdös, Tarski). If $B$ is infinite then

sat $(B) =$ $\kappa^+$ ($\kappa$ infinite) / weakly inaccessible $(> \omega)$ .

Combinatorial facts

A Let $\{X_i , i\in I\}$ be a family of sets. A set $\mathcal{S}\subseteq \prod_{i\in I} X_i$ is called a finitely distingueshed family (FDF) if for any finite $\mathcal{S}_o \subseteq \mathcal{S}$ there is an $i\in I$ such that $|\{f(i) ; f\in \mathcal{S}_o\}| = |\mathcal{S}_o|$ .

L 1 If $X_i$'s are infinite, then there is a FDF $\mathcal{S}\subseteq \prod X_i$ with $|\mathcal{S}| = |\prod X_i|$ .

Consider $B = \mathcal{P}(K)$ for infinite $K$ . We can obtain very easily an independent family $\mathcal{P}_o \subseteq$ Part (B) such that $|\mathcal{P}_o| = \omega$ and $|p| = K$ for $p\in \mathcal{P}_o$ . Using L 1 and $\mathcal{P}_o$ we obtain the well-known fact ([EK],[Ke],[Ku]), namely, there is an independent family of partitions $\mathcal{P}\subseteq$ Part ($\mathcal{P}(K)$) such that $|\mathcal{P}| = 2^K = |B|$ and $(\forall p\in \mathcal{P})$ $|p| = K$ .

Corollary. If $B$ is a cBA and $B = \sum\{B_u , u\in p\}$ and $B_u$'s are semifree then $B$ is semifree, too.

B The following lemma is a straightforward reformulation of a result of Vladimirov and Monk ([V],[M]).

L 2 Let $B$ be a cBA and $\mathcal{P}\subseteq$ Part (B) . For $p\in \mathcal{P}$ let $p^{\Sigma} = \{\bigvee p_1 ; p_1 \subseteq p\}$. Let $(\mathcal{P}^{\Sigma})^{\Pi} = \{\bigwedge a ; a$ is a selector of $\{p^{\Sigma} ; p\in \mathcal{P}\}\}$ .
If for every $u\in \bigcup\{p ; p\in \mathcal{P}\}$ the set $\{x\leq u ; x \in (\mathcal{P}^{\Sigma})^{\Pi} - \{0\}\}$ is not dense in $B_u$ , then there is a partition $q = \{x_o, x_1\}$ such that $x\wedge u \neq 0$ for every $x\in q$ and $u\in \bigcup \mathcal{P}$

C In the sequel we assume that all BA's are homogeneous in sat.

We use the following "disjoint refinement lemma" from [BV] in the proof of L 3. Let $\nu$ be a cardinal, $\nu^+ < \operatorname{sat}(B)$ . Then for any family $\{u_\alpha ; \alpha < \nu\} \subseteq B^+$ there is a disjoint refinement, i.e. a family

$\{v_\alpha \ ; \ \alpha < \nu\} \subseteq B^+$ such that $v_\alpha \leq u_\alpha$ and $v_\alpha \wedge v_\beta = 0$ if $\alpha \neq \beta$ .

L 3 Let sat (B) = K be a weakly inaccessible cardinal. Then there is an independent family $\mathcal{P}$ of partitions on B such that

(i) $|\mathcal{P}| = K$

(ii) $\sup\{|p| \ ; \ p \in \mathcal{P}\} = K$ .

For a proof of the theorem it is sufficient to deal only with atomless cBA's . If B is not atomless then $B = B_1 \oplus B_2$ , where $B_1$ is atomic and $B_2 = 0$ or $B_2$ is atomless. If $|B| = |B_1|$ , B is then semifree because $B_1$ is by the classical result. Otherwise $|B| = |B_2|$ and B is semifree iff $B_2$ is.

Let $B = \sum\{B_u \ ; \ u \in p\}$ be a decomposition of an atomless cBA B into factors homogeneous in the both cardinal characteristics sat and d . Then it is sufficient to prove that $B_u$'s are semifree.

Thus, let B be an atomless cBA homogeneous in sat and d .

Case 1. (Well-known before [Ky])

sat (B) = $K^+$ and d(B) = $\lambda$.

Then $|B| = \lambda^K$ and we can use L 1, L 2 .

Case 2. sat (B) = K , K is weakly inaccess.

d(B) = $\lambda$ .

Then $|B| = \lambda^{\underset{\smile}{K}}$ and we can use L 1, L 2, L 3.

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