Bohuslav Balcar; F. Franěk
Independent families on complete Boolean algebras


Persistent URL: http://dml.cz/dmlcz/701138

Terms of use:

© Institute of Mathematics of the Academy of Sciences of the Czech Republic, 1979

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
We present definitions and lemmas concerning a proof of the following fact, without any set-theoretical assumptions.

**Theorem.** Every infinite complete Boolean algebra contains a free subalgebra of the same cardinality.

This solves the Question 44 of [vD,M,R]. The history of this problem and a survey of partial solutions ([Ko],[Ky],[M]) is given in [Bla].

The theorem extends the classical result of Hausdorff and Pospíšil concerning complete atomic BA's (=$\mathcal{P}(K)$) to arbitrary cBA's.

Let us summarize some well-known consequences of the Theorem. In what follows, $B$ denotes an infinite cBA and $X$ denotes an infinite extremally disconnected compact (e.d.c.) space.

**C1.** Let $\mathcal{U}(B)$ be the set of all ultrafilters on $B$, then

$$\text{card}(\mathcal{U}(B)) = 2^{\text{card}(B)}; \text{ equivalently, } \text{card}(X) = 2^{w(X)},$$

where $w(X)$ is the weight of $X$.

**C2.** There are many (=$|\mathcal{U}(B)|$) ultrafilters on $B$ which have the character (=$\text{the least cardinality of a set of generators}$) equal to $|B|$.

The consequences C1 and C2 solve problems raised by Efimov [Ef].

**C3.** If $C$ is a cBA with $|C| \leq |B|$ then there is a homomorphism $f: B \rightarrow C$; equivalently, for an e.d.c. space $Y$ with $w(Y) \leq w(X)$ there is an embedding of $Y$ into $X$. 
There is a continuous mapping \( f : X \overset{\text{onto}}{\longrightarrow} \{0,1\}^{w(X)} \).

The space \( X \) contains a copy of itself as a nowhere dense subset and therefore \( X \) is not homogeneous. \([F]\).

Notations, definitions

For a BA \( B \) let \( B^+ = B - \{0\} \). For \( u \in B^+ \) let \( B_u \) denote a "partial subalgebra" of \( B \) with the universe \( \{v \leq u : v \in B\} \).

(i) \( \text{Part}(B) = \{ p \leq B^+ ; \forall p = 1 \text{ and the elements of } p \text{ are pairwise disjoint} \} \).

(ii) \( \rho \subseteq \text{Part}(B) \) is called an independent family of partitions if for any finite set of partitions \( \{p_0, \ldots, p_{n-1}\} \leq \rho \) and every mapping \( f : n \rightarrow \bigcup\{p_i , i < n\} \) with \( f(i) \in p_i \) we have \( \bigwedge\{f(i), i < n\} \neq 0 \).

(iii) \( B \) is semifree if there is an independent family of partitions \( \rho \) on \( B \) with \( |\rho| = |B| \).

Hence the theorem is equivalent to the statement "every infinite cBA is semifree".

(iv) \( D \subseteq B^+ \) is dense in \( B \) if \( (\forall v \in B^+)(\exists u \in D) \ u \leq v \);
\[
\text{d}(B) = \min \{ \text{card}(D) ; D \text{ is dense in } B \}.
\]

(v) \( \text{sat}(B) = \min \{\nu ; (\forall p \in \text{Part}(B)) (|p| < \nu)\} \) (\( \nu \) less than)

Trivially, \( \text{sat}(B) \geq \text{sat}(B_u) \), \( \text{d}(B) \geq \text{d}(B_u) \) for \( u \in B^+ \). Hence for a cBA \( B \) there is a partition \( p \) such that
\[
B = \sum_{u \in p} B_u \quad \text{(a product in the category of BA's)} \text{ and all } B_u's \text{ are homogeneous in sat and } d.\]

(vi) (Erdős, Tarski). If \( B \) is infinite then
\[
\text{sat}(B) = \begin{cases} K^+ & (K \text{ infinite}) \\ \text{weakly inaccessible} & (> \omega) \end{cases}
\]
Combinatorial facts

A Let \( \{X_i, i \in I\} \) be a family of sets. A set \( \mathcal{Y} \subseteq \prod_{i \in I} X_i \) is called a finitely distinguished family (FDF) if for any finite \( \mathcal{Y}_0 \subseteq \mathcal{Y} \) there is an \( i \in I \) such that
\[
| \{ f(i) : f \in \mathcal{Y}_0 \} | = | \mathcal{Y}_0 | .
\]

L 1 If \( X_i \)'s are infinite, then there is a FDF \( \mathcal{Y} \subseteq \prod_{i \in I} X_i \) with
\[
| \mathcal{Y} | = | \prod_{i \in I} X_i | .
\]

Consider \( B = \mathcal{P}(K) \) for infinite \( K \). We can obtain very easily an independent family \( \mathcal{P}_0 \subseteq \text{Part} \ (B) \) such that
\[
| \mathcal{P}_0 | = \omega \quad \text{and} \quad | p | = K \quad \text{for} \quad p \in \mathcal{P}_0 .
\]
Using L 1 and \( \mathcal{P}_0 \) we obtain the well-known fact \( ([\mathcal{E}K],[\mathcal{K}e],[\mathcal{K}u]) \), namely, there is an independent family of partitions \( \mathcal{J} \subseteq \text{Part} \ (\mathcal{P}(K)) \) such that
\[
| \mathcal{J} | = 2^K = | B | \quad \text{and} \quad (\forall p \in \mathcal{P}) \ | p | = K .
\]

Corollary. If \( B \) is a cBA and \( B = \sum \{B_u, u \in p\} \) and \( B_u \)'s are semifree then \( B \) is semifree, too.

\[B\]

The following lemma is a straightforward reformulation of a result of Vladimirow and Monk \( ([V],[M]) \).

L 2 Let \( B \) be a cBA and \( \mathcal{P} \subseteq \text{Part} \ (B) \). For \( p \in \mathcal{P} \) let
\[
\mathcal{P}^\Sigma = \{ \forall p_1 ; p_1 \leq p \} .
\]
Let \( (\mathcal{J}^\Sigma)^\Pi = \{ \forall a ; a \text{ is a selector of} \ \{ p \in \mathcal{P}^\Sigma ; p \in \mathcal{J} \} \} .
\]

If for every \( u \in \bigcup \{ p ; p \in \mathcal{P} \} \) the set \( \{ x \leq u ; x \in (\mathcal{J}^\Sigma)^\Pi - O \} \) is not dense in \( B_u \), then there is a partition
\[q = \{ x_0, x_1 \} \] such that \( x \land u \not\in O \) for every \( x \in q \) and \( u \in \bigcup \mathcal{P} \).

C In the sequel we assume that all \( \mathcal{A} \)'s are homogeneous in \( \text{sat} \).

We use the following "disjoint refinement lemma" from \([\mathcal{B}V]\) in the proof of L 3. Let \( \nu \) be a cardinal, \( \nu^+ < \text{sat} (B) \). Then for any family \( \{ u_\alpha ; \alpha < \nu \} \subseteq B^+ \) there is a disjoint refinement, i.e. a family
\( \{ v_\alpha : \alpha < \nu \} \subseteq B^+ \) such that \( v_\alpha \leq u_\alpha \) and \( v_\alpha \land v_\beta = 0 \) if \( \alpha \neq \beta \).

L 3 Let \( \text{sat}(B) = K \) be a weakly inaccessible cardinal. Then there is an independent family \( \mathcal{I} \) of partitions on \( B \) such that

1. \( |\mathcal{I}| = K \)
2. \( \sup \{|p| ; p \in \mathcal{I}\} = K \).

For a proof of the theorem it is sufficient to deal only with atomless cBAs. If \( B \) is not atomless then \( B = B_1 \oplus B_2 \), where \( B_1 \) is atomic and \( B_2 = 0 \) or \( B_2 \) is atomless. If \( |B| = |B_1| \), \( B \) is then semifree because \( B_1 \) is by the classical result. Otherwise \( |B| = |B_2| \) and \( B \) is semifree iff \( B_2 \) is.

Let \( B = \sum \{ B_u ; u \in \mathcal{P} \} \) be a decomposition of an atomless cBA \( B \) into factors homogeneous in the both cardinal characteristics \( \text{sat} \) and \( d \). Then it is sufficient to prove that \( B_u \)'s are semifree.

Thus, let \( B \) be an atomless cBA homogeneous in \( \text{sat} \) and \( d \).

Case 1. (Well-known before \([\text{Ky}]\))

\( \text{sat}(B) = K^+ \) and \( d(B) = \lambda \).

Then \( |B| = \lambda^K \) and we can use L 1, L 2.

Case 2.

\( \text{sat}(B) = K \), \( K \) is weakly inaccessible.

\( d(B) = \lambda \).

Then \( |B| = \lambda^K \) and we can use L 1, L 2, L 3.
References

[BV] B. B. car, P. Vo- J. Defining systems on Boolean alg br
L c u t e te n Math. 619, 5-5

[Bla] A. Bialszyń: O napur, ggs of extremally diccon ect ed comp-
act spaces onto Cantor abhs, Proceedings of Collo-
quium on Topology (Budapest 1978) (to appear)


(Russian)


[H] F. Hausdorff: Ober zwei Sätze von G. Fichtenholz und L. Kantorowich, Studia Math. 6 (1936), 18-19


[Ky] S. Keslyakov: Free subalgebras of complete Boolean algeb-
ras and spaces of continuous functions, Sibirski Mat. Zh. 14 (1973), 569-581

[Ke] J. Ketonen: Everything you wanted to know about ultrafil-
ters - ..., Doctoral dissertation University of Wis-
consin 1971

[Ko] S. Koppelberg: Free subalgebras of complete Boolean algeb-
ras, Notices Amer. Math. Soc. 20 (1973), A-418


[M] I.D. Monk: On free subalgebras of complete Boolean alge-
bras, Arch. der Math. 29 (1977), 113-115

(1937), 845-846


Dept. OR, ČKD - Polovodiče, 140 03 Prague, Czechoslovakia
Dept. of Math. University of Toronto, Toronto, Ontario, Canada