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A generalization of component categories


Persistent URL: http://dml.cz/dmlcz/701139

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Component categories have been investigated by several authors (see [2], [4]), for topological functors $G : X \to \text{Ens}$, where $\text{Ens}$ is the category of sets. We give a generalization to arbitrary functors $G : X \to \text{Ens}$, following an idea of Pumplün and Holmann (unpublished). and we are led to a generalization of a Galois correspondence given by Maranda. [3]. The results are part of my thesis.

**Theorem**: Let $G : X \to \text{Ens}$ be a functor, and $A \subseteq X$ a full subcategory such that $G|A$ is pointwise non-void, i.e. $G(A) \neq \emptyset$ for all $A \in \text{Ob}(A)$. Then there is a functor $Q : X \to \text{Ens}$ and a natural transformation $\xi_{G,A} : G \to Q_{G,A}$ with the following properties:

(i) For all $A \in \text{Ob}(A)$ the cardinality of $G(A)$ is 1.

(ii) If $\alpha : G \to P$ is a natural transformation, such that $\alpha(A)$ is of cardinality 1 for all $A \in \text{Ob}(A)$, then there is a unique natural transformation $\xi : Q_{G,A} \to P$ with $\xi_{Q_{G,A}} = \alpha$.

(iii) For all $X \in \text{Ob}(X)$, $\xi_{G,A}(X)$ is onto.

**Definition**: Let $\alpha : G \to P$ be a natural transformation, such that $\alpha(X)$ is onto for all $X \in \text{Ob}(X)$. Then $\text{Connect}(\alpha)$ denotes the full subcategory of $X$ generated by all $X$-objects $A$ where $P(A)$ is a singleton.

Now the above theorem can be interpreted as a Galois adjunction. between $\text{Connect}$ and $\xi_{G,-}$, considered as meta-functors between the meta-category of all full subcategories $A$ of $X$ with $G|A$ pointwise non-void and the meta-category of all pointwise surjective natural transformations with domain $G$. 

A GENERALIZATION OF COMPONENT CATEGORIES

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Definition: Let \( G : \mathcal{A} \rightarrow \mathsf{Ens} \) be a functor. A full subcategory \( \mathcal{A} \subseteq \mathcal{X} \) is called a G-component category, iff \( \mathcal{A} = \text{Conn\&at}(\mathcal{A}) \) (or, equivalently, iff there exist a functor \( \alpha \) and a natural transformation \( \alpha : G \rightarrow \mathcal{P} \), such that \( \mathcal{A} \) is the full subcategory generated by all objects \( A \in \text{Ob}(\mathcal{X}) \) where \( \mathcal{P}(A) \) is a singleton).

Corollary: If \( \mathcal{A} \subseteq \mathcal{X} \) is a full subcategory and \( G : \mathcal{X} \rightarrow \mathsf{Ens} \) is a functor with \( G \mathcal{A} \) pointwise non-void, then \( \text{Conn\&at}(\mathcal{A}) \) is the smallest G-component category containing \( \mathcal{X} \).

Theorem: Let \( G : \mathcal{X} \rightarrow \mathsf{Ens} \) be a mono-fibration (i.e. any injective map \( m : U \rightarrow G(X) \) has an initial lifting to an \( \mathcal{X} \)-morphism \( \bar{m} : \overline{U} \rightarrow X \)). Let \( \mathcal{M} \) denote the class of all \( G \)-initial liftings of injective maps and let \( \mathcal{A} \subseteq \mathcal{X} \) be a full and replete subcategory with \( G \mathcal{A} \) pointwise non-void. Let \( \mathcal{A} \) contain all \( A \in \text{Ob}(\mathcal{X}) \) for which \( \mathcal{P}(A) \) is a singleton. Then the following statements are equivalent:

(i) \( \mathcal{A} \) is a G-component category.

(ii) \( \mathcal{A} \) is strongly locally \( \mathcal{M} \)-coreflective in the sense of [1], i.e. for any \( x \in \text{Ob}(\mathcal{X}) \) there is a family \( (u_i : Z_i \rightarrow X)_{i \in I} \), all \( Z_i \) are in \( \mathcal{A} \), such that for any \( f : A \rightarrow X \) with \( A \in \text{Ob}(\mathcal{A}) \) there is a unique pair \((i, h)\) with \( i \in I \), \( h : A \rightarrow Z_i \), and \( u_i h = f \).

(iii) \( \mathcal{A} \) fulfills the following conditions:

1) If \( A \in \text{Ob}(\mathcal{A}) \), \( f : A \rightarrow B \) is an \( \mathcal{X} \)-morphism, \( G(f) \) is onto, then \( B \in \text{Ob}(\mathcal{A}) \).

2) Let \( X \in \text{Ob}(\mathcal{X}) \), \( (m_i : A_i \rightarrow X)_{i \in I} \), \( I \neq \emptyset \) be a family of \( G \)-initial morphisms, such that \( G(m_i) \) is onto one for all \( i \in I \). If now \( \cap \{G(m_i)[A_i]\} \neq \emptyset \), \( \cup \{G(m_i)[A_i]\} = G(X) \) then \( X \in \text{Ob}(\mathcal{A}) \).

This characterization leads to a general investigation of full replete strongly coreflective subcategories of an arbitrary category.
Definition: Let $X$ be a category.

(i) If $A \in \text{Ob}(X)$, $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$ is a sink, $A$ is called locally uniquely projective with respect to $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$, iff for any $X$-morphism $f : A \to Y$ there is a unique pair $(i, h)$ with $i \in I$, $h_m = f$. Equivalently, we say $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$ is locally uniquely coextendable with respect to $A$.

(ii) If $A \subseteq X$ is a full subcategory let $P^\text{loc}_1(A)$ denote the conglomerate of all locally coextendable sinks with respect to all $A \in \text{Ob}(A)$.

(iii) If $S$ is a conglomerate of $X$-sinks, let $C^\text{loc}_1(S)$ denote the full subcategory of $X$ generated by all locally uniquely projective objects with respect to all $S$-sinks.

(iv) If $M$ is a class of $X$-morphisms, let $\wedge M$ denote the conglomerate of all sinks $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I})$ with $m_i \in M$ for all $i \in I$.

(v) Let $A$ be a full replete subcategory of $X$ and $M$ a class of morphisms. $A$ is called strongly locally $M$-coreflective, if for any $Y \in \text{Ob}(X)$ there is a sink $(Y, (X_i \xrightarrow{m_i} Y)_{i \in I}) \in P^\text{loc}_1(A) \cap \wedge M$ with $X_i \in \text{Ob}(A)$ for all $i \in I$. $A$ is called strongly locally $X$-coreflective, iff $A$ is strongly locally $X$-coreflective.

As $P^\text{loc}_1$ and $C^\text{loc}_1$ form a Galois correspondence, we look at the full subcategories closed under the correspondence. We get the following
Theorem: Let $X$ be a category, $\mathcal{A} \subseteq X$ a full subcategory, $\mathcal{M}$ a class of $X$-morphisms.

(i) If $\mathcal{A} = \mathcal{F}_{\text{loc}}(\mathcal{C})_{\mathcal{F}_{\text{loc}}}(\mathcal{A})$, then $\mathcal{A}$ is closed under the formation of connected colimits.

(ii) If $\mathcal{A}$ is strongly locally $\mathcal{M}$-coreflective, then $\mathcal{A} = \mathcal{F}_{\text{loc}}(\mathcal{C})_{\mathcal{F}_{\text{loc}}}(\mathcal{A}) \cap \mathcal{M}$.

(iii) If $\mathcal{A}$ has locally coorthogonal $(\mathcal{F}, \mathcal{M})$-factorizations (see [5]), then $\mathcal{A} = \mathcal{F}_{\text{loc}}(\mathcal{C})_{\mathcal{F}_{\text{loc}}}(\mathcal{A}) \cap \mathcal{M}$ implies that $\mathcal{A}$ is strongly locally $\mathcal{M}$-coreflective.

References:


